02: Differential Equations & Domains

15-424: Foundations of Cyber-Physical Systems

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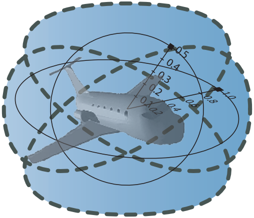

Outline

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1. Introduction
2. Differential Equations
3. Examples of Differential Equations
4. Domains of Differential Equations

Differential Equations as Models of Continuous Processes

Example (Vector field and one solution of a differential equation)

$$\begin{bmatrix} y'(t) = & f(t,y) \\ y(t_0) = & y_0 \end{bmatrix}$$

Intuition:

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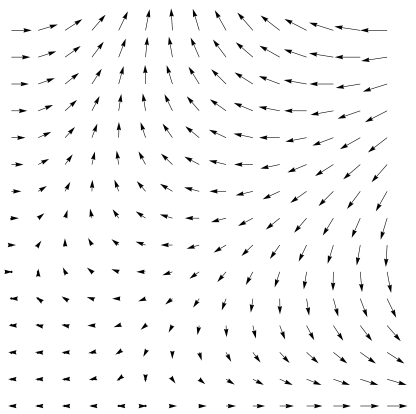
1. At each point in space, plot the value of $f(t, y)$ as a vector



Example (Vector field and one solution of a differential equation)

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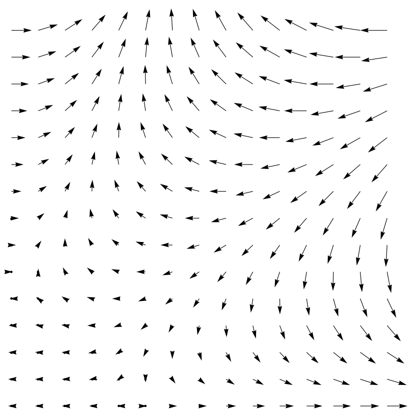
2. Start at initial state y_0 at initial time t_0



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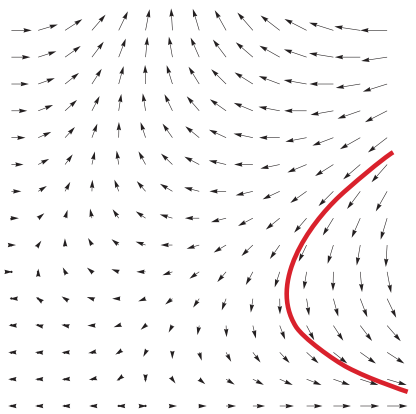

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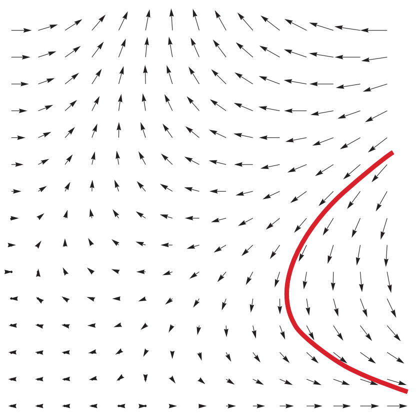

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↯ The diagram should show infinitely many vectors . . .

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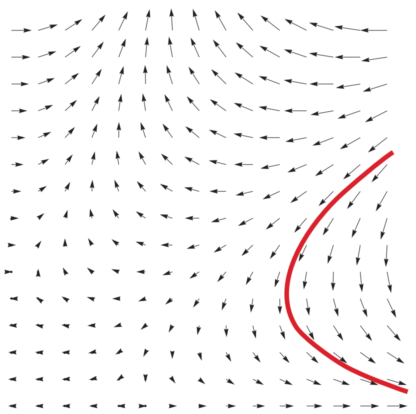

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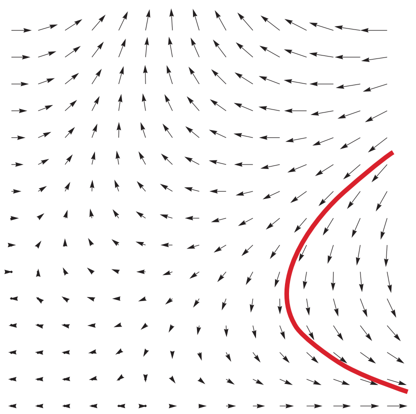

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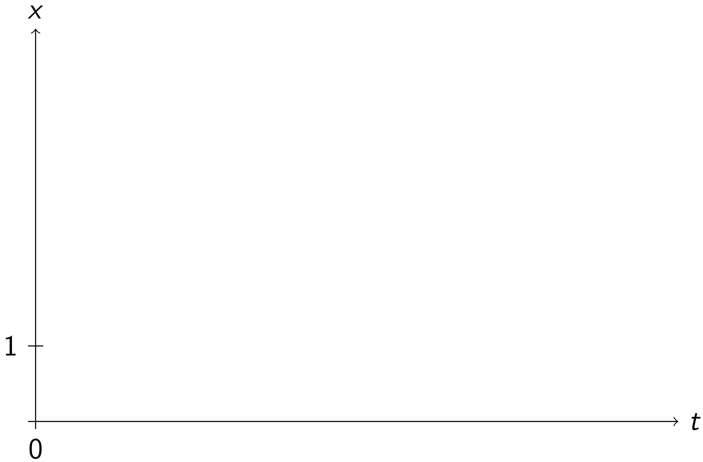
Well it's a wee bit more complicated

Intuition for Differential Equations



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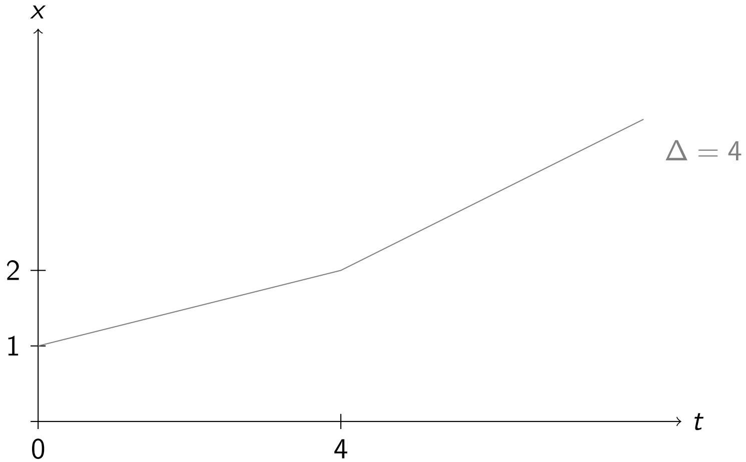

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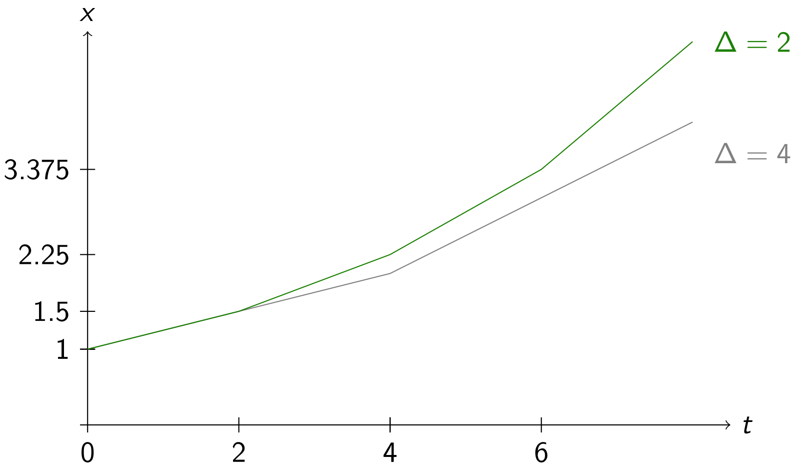

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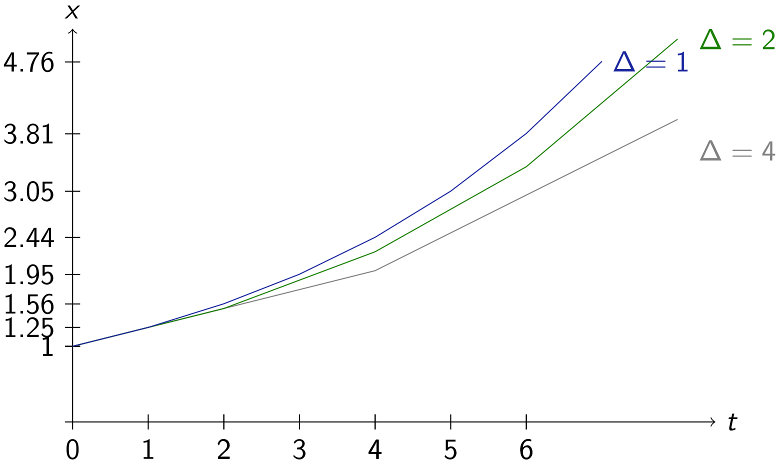

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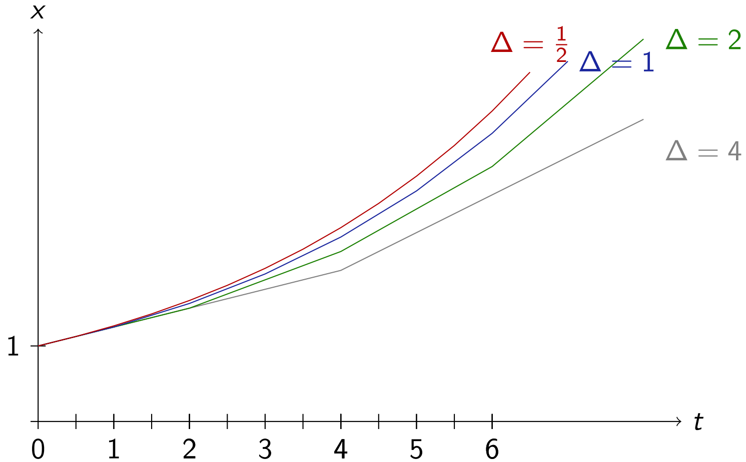

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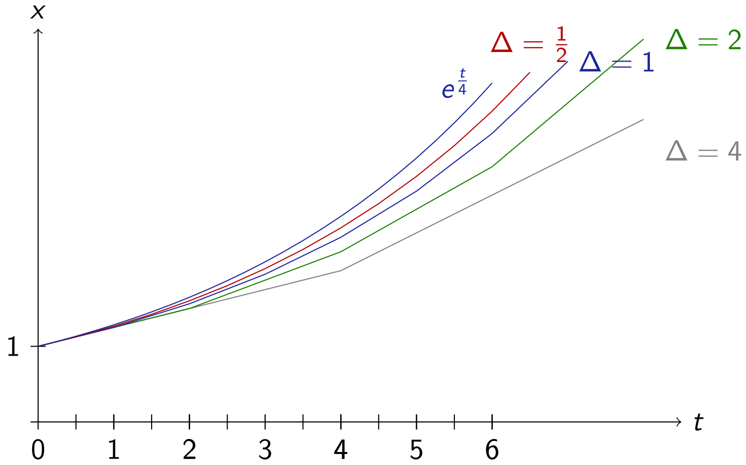

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Outline

The Meaning of Differential Equations

1. What exactly is a vector field?
2. What does it mean to describe directions of evolution at every point in space?
3. Could directions possibly contradict each other?

Importance of meaning

The physical impacts of CPSs do not leave much room for failure, so we immediately want to get into the mood of consistently studying the behavior and exact meaning of all relevant aspects of CPS.

Differential Equations & Initial-Value Problems

Definition (Ordinary Differential Equation, ODE)

$f : D \to \mathbb{R}^n$ on domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \to \mathbb{R}^n$ is *solution* of initial value problem (IVP)

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on interval $I \subseteq \mathbb{R}$, iff, for all times $t \in I$,

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.

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If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$.
If f continuous, then Y continuously differentiable.

Outline

Example: A Constant Differential Equation

Example (Initial value problem)

$$\begin{bmatrix} x'(t) = & 5 \\ x(0) = & 2 \end{bmatrix}$$

has a solution

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has a solution $x(t) = 5t + 2$

Check by inserting solution into ODE+IVP.

$$\left[\begin{array}{rl} (x(t))' = & (5t+2)' = 5 \\ x(0) = & 5 \cdot 0 + 2 = 2 \end{array}\right]$$

□

Example: A Linear Differential Equation from before

Example (Initial value problem)

$$\begin{bmatrix} x'(t) = & \frac{1}{4}x(t) \\ x(0) = & 1 \end{bmatrix}$$

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ODE Examples

ODE	Solution
$x' = 1, x(0) = x_0$	$x(t) = x_0 + t$

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$x' = \frac{1}{x}, x(0) = 1$	$x(t) = \sqrt{1 + 2t} \dots$

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$x'(t) = e^{t^2}$	non-elementary

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$x'(t) = e^{t^2}$	non-elementary

Takeaway Message

Descriptive power of differential equations

1. Solutions of differential equations can be much more involved than the differential equations themselves.
2. Representational and descriptive power of differential equations!
3. Simple differential equations can describe quite complicated physical processes.
4. Local description as the direction into which the system evolves.

Outline

Evolution Domain Constraints

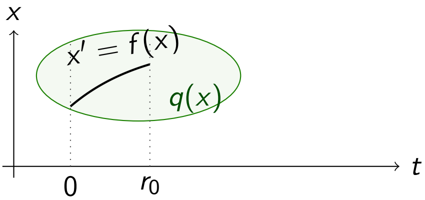
Enable Cyber to interact with Physics

Definition (Evolution domain constraints)

A *differential equation* $x' = f(x)$ *with evolution domain* $q(x)$ is denoted by

$$x' = f(x) \,\&\, q(x)$$

conjunctive notation (&) signifies that the system obeys the differential equation $x' = f(x)$ and the evolution domain $q(x)$.



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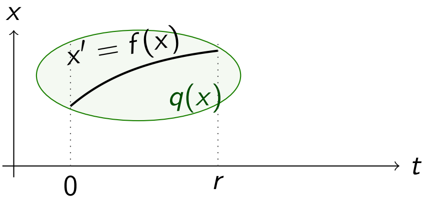

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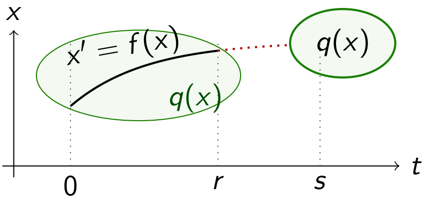
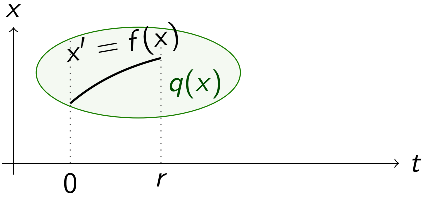


Semantics of ODE with Evolution Constraints

Definition (Semantics of differential equations)

A function $\varphi : [0, r] \to \mathcal{S}$ of some duration $r \geq 0$ satisfies the differential equation $x' = f(x) \,\&\, q(x)$, written $K, \varphi \models x' = f(x) \land q(x)$, iff:

1. $\varphi(\zeta)(x') = \frac{d\varphi(t)(x)}{dt}(\zeta)$ exists at for all times $0 \leq \zeta \leq r$
2. $\varphi(\zeta) \in [\![x' = f(x) \land q(x)]\!]$ for all times $0 \leq \zeta \leq r$



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