15-424/15-624 Recitation 10
Assignment review and virtual substitution

1. **Assignment review: derivatives**

   We quickly went over the $(\theta < \eta)' \equiv (\theta' \leq \eta')$ proof. No online solutions can be made available, but the idea is that you have you either use a technique similar to the one from lectures with the mean value theorem, or be clever!

   An important insight is which direction of the equivalence is soundness and which is completeness. Can you figure it out?

2. **Assignment review: non-deterministic assignment**

   We also covered non-deterministic assignment, and whether it brings us some new tools for designing controllers, in particular w.r.t. the how general we can make our controllers, and thus, our proofs.

3. **Virtual substitution**

   Annika presented some examples! Virtual substitution works for a fairly small subset of formulas, which contain only polynomials of degree 2 or less. The idea is that roots of existentially quantified quadratic polynomials can be replaced by the quadratic equation itself (as long as some obvious conditions are met).

   Also important is to realise that the quadratic equation is a *square root expression* (SRE) of the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are polynomials of rational coefficients (since that's what we can write in $\mathsf{d}\mathcal{L}$. These SREs are closed under addition and multiplication, meaning that if you add or multiply them together, you end up with another SRE! This turns out to be pretty important! Incidentally, polynomials are also SREs!

   To apply virtual substitution, the following steps are used:

   (a) Take your formula and place it in canonical form using the normalisation and substitutions from pages L18.7-L18.9[1].

   (b) You should end up with existential quantifiers over the roots of single-variable polynomials of degree two or less, i.e. $\exists x. ax^2 + bx + c = 0$, and with one or more formulas about SREs, such as $s \sim 0$, for $\sim \in \{=, \neq, <, \leq, >, \geq\}$. Here, s will initially be a polynomial, but will change into an SRE later.

   (c) Choose one of the existential quantifiers $\exists x. p(x) = 0$, and name the rest of the formula $F(x)$. Notice how x can occur in $F(x)$. Apply Theorem 5 on L18.12. This eliminates the existential quantifier over x, and tells you to substitute x for the solution to the quadratic equation, which is an SRE. This means that F will be potentially composed of SREs instead of polynomials.

[1] `http://symbolaris.com/course/fcps14/18-virteq.pdf`

(d) Notice that in $\text{FOL}_{\mathbb{R}}$, you can't *actually* substitute in the quadratic equation solution because it contains square roots, and those aren't understood by $\text{FOL}_{\mathbb{R}}$. Nonetheless, let's break out of $\text{FOL}_{\mathbb{R}}$ and into the virtual realm of SREs (hence virtual substitution, get it? So exciting!)... Since you're replacing a variable in a polynomial with a square root expression, the result is a square root expression!

(e) Keep applying Theorem 5 to all the quantifiers and performing all the resulting virtual substitutions into those SREs.

(f) Once all quantifiers are eliminated, you end up with a formula containing lots of quantifier-free SREs $sre \sim 0$. *But wait, there's more!*

(g) We can't stop now, since SREs can contain square-roots, and $\text{FOL}_{\mathbb{R}}$ has a pretty severe square-root allergy, *and* it forgot to brings its epi-pen with it (oh, $\text{FOL}_{\mathbb{R}}$, you so silly!). How can we save it from the impending logico-medical emergency? It turns out that there are some great $\sqrt{\ }$-free equivalences for SRE expressions, which you can find in L18.14. So all you need to do is apply those equivalences, and end up with a wonderful quantifier-free, $\sqrt{\ }$-free formula which is muuuuuch easier to check!