Assignment 5: Differential Auxiliaries, dTL and Quantifier Elimination
15-424/15-624 Foundations of Cyber-Physical Systems
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Due: **Beginning of recitation**, Friday 11/7/14
Total Points: 60

1. **Valid, Satisfiable, or Unsatisfiable.** Determine whether each of the following *differential temporal dynamic logic* (dTL) formulas is valid, satisfiable, or unsatisfiable. Briefly justify.

   (a) $(x \leq 0 \land v \geq 0 \land T > 0) \to [t := 0, x' = v, v' = a, t' = 1; ?t = T; ?x < 0]\Box x < 0$

   (b) $\big([a := A; a := -B; v' = a]\ v \geq 0\big) \leftrightarrow \big([a := A; a := -B; v' = a]\ \Box v \geq 0\big)$

   (c) $\big([a := A; a := -B; v' = a]\ a \leq 0\big) \leftrightarrow \big([a := A; a := -B; v' = a]\ \Box a \leq 0\big)$

   (d) $\big([x' = t, t' = 1\ \&\ t \geq 0]\Box\ x \geq 0\big) \leftrightarrow \big([x' = t, t' = 1; ?t \geq 0]\ x \geq 0\big)$

2. **Quantifier Elimination.** Apply quantifier elimination to eliminate the quantified variables in each of the following formulas. You must use virtual substitution for this question.

   (a) $\exists x\ (x^2 + 4x + 4 = 0 \land y > x)$

   (b) $\exists x\ (2x^2 - 7x + 15 = 0 \land y \leq x^2 \land y + x = 0)$

   (c) $\exists x\ (x^2 - 10x + 10 \leq 0 \land y - x \geq 0)$

3. **d$\mathcal{L}$ vs. dTL.** Consider the formula $F \equiv [\alpha]\phi \leftrightarrow [\alpha]\Box\phi$. In this exercise we will explore when this equivalence holds.

   (a) Let α be a simple assignment. Then

   $$F \equiv [x := \theta]\phi(x) \longleftrightarrow [x := \theta]\Box\phi(x)$$

   For what choice of θ is F necessarily true? Is it necessarily true for any other choice? Very briefly justify.

   (b) What if we slightly change the program to be *extra safe*, as follows:

   $$F \equiv [x := \theta; ?\phi(x)]\phi(x) \longleftrightarrow [x := \theta; ?\phi(x)]\Box\phi(x)$$

   Does F hold now? Why?

   (c) Let's not forget our best friends, the ODEs.

   $$F \equiv [x' = \theta]\phi(x) \longleftrightarrow [x' = \theta]\Box\phi(x)$$

   For what choice of θ is F necessarily true? Can you modify the program without specifying θ so that F is necessarily true?

(d) Define a new grammar (e.g. $\alpha, \beta ::= \alpha \cup \beta \mid \alpha; \beta \mid ...$), specifying what hybrid programs can be written, that ensures that the property $[x := \theta]\phi(x) \longleftrightarrow [x := \theta]\Box\phi(x)$ holds. *Hint:* you have access to the property $\phi(x)$ and can use it in the grammar. You should try restricting the original hybrid program grammar.

4. **Convergence and Divergence.** Consider the infinite summation over function $f(n)$, with $f(n) \geq 0$ for all $n \in \mathbb{N}$:

$$\sum_{n=0}^{\infty} f(n)$$

(a) Write a d$\mathcal{L}$ formula, containing one [] or $\langle\rangle$ modality, in d$\mathcal{L}$, which, if proved valid, would guarantee the sum converges.

(b) Write a d$\mathcal{L}$ formula, containing one [] or $\langle\rangle$ modality, in d$\mathcal{L}$, which, if proved valid, would guarantee the sum diverges.

Hint: for a refresher, `http://en.wikipedia.org/wiki/Convergent_series`

5. **Chord length vs. arc length.** Because you've already done lab 3, we know you know how fun circular dynamics are! So in this exercise and the following, we are going to prove yet *another* interesting property of circular dynamics! Let's dive right in!

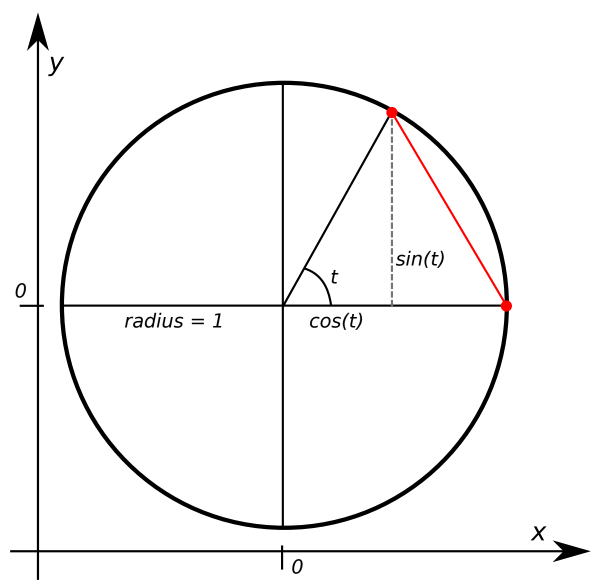

Figure 1: A beautiful picture, meticulously & lovingly hand-crafted by your local TAs

The following formula states that chord length (in red) is always smaller than arc length in a circle of radius 1.

$$x = 1 \wedge y = 0 \wedge t = 0 \rightarrow [x' = -y, y' = x, t' = 1]\ 2(1 - x) \leq t^2$$

It will be your job to figure out why that's the case! Fortunately, your dearest TAs are here to help you figure out how the heck $2(1-x) \leq t^2$ means "the chord length is less than the arc length".

Despite its misleading name, t is not time (sneaky, sneaky t!). As you can see above, it's the angle. Moreover, we know that $t = \frac{\text{arc length}}{r}$, i.e. arc length $= tr$. One famous version of this formula is the arc of 360° degrees, i.e. $c = 2\pi r$. Because we are assuming the unit radius, then the arc is simply t!

Using the quantities in Figure 1, and the initial values of $x = 1, y = 0$, derive and explain how $2(1-x) \leq t^2$ means that chord length is smaller than arc length.

6. **Chord length vs. arc length (proof edition)!** We know you like circular motion. But we're on to your dark secret! More than circular motion, we know you love *proving things*! What can we do but oblige?

   Please prove the formula in d$\mathcal{L}$.

   Hint: use DC and DI, a dash of DW, and a sprinkle of circular motion properties. One very useful insight is that applying DI will generally yield formulas that you need in order to prove the original property. So try to find a way to include those formulas in the proof!