Assignment 1: Introduction to Hybrid Programs
15-424/15-624 Foundations of Cyber-Physical Systems
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Total Points: 60

1. **Term, formula, hybrid program, or none of the above?**
   For each of the following, determine if the expression is a d$\mathcal{L}$ term, a well-formed d$\mathcal{L}$ formula, a well-formed hybrid program, or none of the above (ie. it is not well-formed). In the case that the expression is none of the above, give a short explanation.

   (a) $z := x^5$

   (b) $?(x > \frac{3}{4})$

   (c) x

   (d) $\forall x \exists y\ (x < y)$

   (e) $z := \pi$

   (f) $(x > y \land (y > z \cup y > w))$

   (g) $(x := x + 1)^*$

   (h) $x \geq y \rightarrow y \geq x$

   (i) 42

   (j) $[?x > 5; x' = y \ \&\ x < 10](y < z \rightarrow y^2 + z^2 < x^2) \rightarrow [x := \cos z](-1 \leq y \land y \leq 1)$

   (k) $[g := 42]$

   (l) $?x := 5$

   (m) $(x > y)^*; ?(x > y)$

   (n) $x := y \cup x := z$

   (o) $x := 0 \rightarrow [x' = -y, y' = x \ \&\ y \leq \frac{1}{2}]x \geq \frac{1}{2}$

   (p) $[x = y; y = z]?(x \geq z)$

   (q) $y > 1 \land z > 1 \rightarrow [x := z](x^y > z)$

2. **Non-Deterministic Evolution**

   $$\beta \equiv \quad x := x_0;\ v := v_0;\ t := 0;\ (x' = v, v' = a, t' = 1 \ \&\ v \geq 0); ?(v = 0)$$

   (a) Assume that $a < 0 \land v_0 \geq 0$. At the end of a run of hybrid program β, what is the value of t as a function of x_0, v_0, and a?

   (b) Suppose we remove the guard $?(v = 0)$ at the end of hybrid program β. Again assuming that $a < 0 \land v_0 \geq 0$, what are the possible values of v at the end of any run of this modified version of β?

   What about the possible values of t?

(c) Suppose we assume instead that $a < 0 \wedge v_0 \leq 0$ (v_0 is **less than** or equal to zero). What are the possible values of v and t at the end of any run of β?

3. **Practice writing hybrid programs.**

   (a) Non-deterministic Choice: Write a hybrid program that assigns either a or b to the variable x.

   (b) The if-then-else construct is not actually in the grammar of hybrid programs; it's just syntactic sugar. Write an equivalent hybrid program which does not depend on if-then-else:

$$\texttt{if } (y^2 \geq 55 \wedge y \leq 0) \; y := y^2 \;\; \texttt{else} \; y := y + 5$$

4. **Safety and Contracts**

   (a) A *safety* property is something that a cyber-physical system should *always* maintain. Describe a cyber-physical system and then name three safety properties that it should never violate.

   (b) Suppose you want to prove that a property ϕ is a safety property of hybrid program α, i.e. that ϕ holds under all possible runs of hybrid program α. Write the $\mathsf{d}\mathcal{L}$ formula that expresses this.

   (c) Now, suppose you want to prove that, given initial conditions ψ, another property ϕ is a safety property of hybrid program α. Write the $\mathsf{d}\mathcal{L}$ formula that expresses this.

   (d) Consider hybrid program α:

$$\alpha \equiv x := v; x := w; ((y := 0; z := 2) \cup (z := 1; y := w - 5)); (x' = 3); ?(y \leq 42)$$

   For each variable in α, list the set of all values they can reach at the end of a run of α.

   (e) Write a first-order logic formula that describes your solution from part 4d, i.e. accurately describes all possible values for each variable at the end of the execution of α.