

# Lecture Notes on Virtual Substitution & Real Arithmetic

André Platzer

Carnegie Mellon University  
Lecture 19

## 1. Introduction

Reasoning about cyber-physical systems and hybrid systems requires understanding and handling their real arithmetic, which can be challenging, because cyber-physical systems can have complex behavior. Differential dynamic logic and its proof calculus [Pla08, Pla10, Pla12] reduce the verification of hybrid systems to real arithmetic. How arithmetic interfaces with proofs has already been discussed in [Lecture 9 on Proofs & Arithmetic](#). How real arithmetic with linear and quadratic equations can be handled by virtual substitution has been shown in [Lecture 18 on Virtual Substitution & Real Equations](#). Today's lecture shows how virtual substitution for quantifier elimination in real arithmetic extends to the case of linear and quadratic inequalities.

These lecture notes are loosely based on [Wei97, Pla10, Appendix D]. They add substantial intuition and motivation that is helpful for following the technical development. More information about virtual substitution can be found in the literature [Wei97]. See, e.g., [PQR09, Pas11] for an overview of other techniques for real arithmetic.

The most important learning goals of this lecture are:

**Modeling and Control:** This lecture refines the indirect impact that the previous lecture had on CPS models and controls by informing the reader about the consequences of the analytic complexity resulting from different arithmetical modeling tradeoffs. There are subtle analytic consequences from different arithmetic formulations of similar questions that can have an impact on finding the right tradeoffs for expressing a CPS model. A safe distance of car  $x$  to a stopping light  $m$  could equally well be captured as  $x \leq m$  or as  $x < m$ , for example.

**Computational Thinking:** The primary purpose of today's lecture is to understand how arithmetical reasoning, which is crucial for CPS, can be done rigorously and

automatically not just for the equations considered in [Lecture 18 on Virtual Substitution & Real Equations](#) but also for inequalities. While formulas involving sufficiently many quadratic equations among other inequalities could already be handled using the techniques from [Lecture 18](#), such extensions are crucial for proving arithmetic formulas that involve only inequalities, which happens rather frequently in a world of CPS where many questions concern inequality bounds on distances. Developing an intuition for the working principles of real arithmetic decision procedures can be very helpful for developing strategies to verify CPS models at scale. We will again see the conceptually very important device in the logical trinity: the flexibility of moving back and forth between syntax and semantics at will. Virtual substitutions will again allow us to move back and forth at will between syntax and semantics. This time, square roots will not be all there is to it, but the logical trinity will lead us to ideas from nonstandard analysis to bridge the gap between semantic operations that are inexpressible otherwise in first-order logic of real arithmetic.

**CPS Skills:** The author is not aware of any impact that this lecture has on CPS skills, because this lecture focuses on the handling arithmetic in CPS analysis.

## 2. Recall: Square Root $\sqrt{\cdot}$ . Substitutions for Quadratics

Recall the way to handle quantifier elimination for linear or quadratic equations from [Lecture 18 on Virtual Substitution & Real Equations](#) by virtually substituting in its symbolic solutions  $x = -c/b$  or  $x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$ , respectively:

**Theorem 1** (Virtual substitution of quadratic equations). *For a quantifier-free formula  $F$  with  $x \notin a, b, c$ , the following equivalence is valid over  $\mathbb{R}$ :*

$$\begin{aligned}
 a \neq 0 \vee b \neq 0 \vee c \neq 0 &\rightarrow \\
 &\left( \exists x (ax^2 + bx + c = 0 \wedge F) \leftrightarrow \right. \\
 &\quad a = 0 \wedge b \neq 0 \wedge F_{\hat{x}}^{-c/b} \\
 &\quad \left. \vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge \left( F_{\hat{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\hat{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)} \right) \right) \\
 &\quad (1)
 \end{aligned}$$

When using virtual substitutions of square roots from [Lecture 18](#), the resulting formula on the right-hand side of the biimplication is quantifier-free and can be chosen for QE( $\exists x (ax^2 + bx + c = 0 \wedge F)$ ) as long as it is not the case that  $a = b = c = 0$ . In case  $a = b = c = 0$ , another formula in  $F$  needs to be considered for directing quantifier elimination by commuting and reassociating  $\wedge$ , because the equation  $ax^2 + bx + c = 0$  is noninformative if  $a = b = c = 0$ , e.g. when  $a, b, c$  are the zero polynomials or even if they just have a common root.

The equivalent formula on the right-hand side of the biimplication in (1) is a formula in the first-order logic of real arithmetic when using the virtual substitution of square root expressions defined in [Lecture 18](#).

### 3. Infinity $\infty$ Substitution

Theorem 1 addresses the case where the quantified variable occurs in a linear or quadratic equation, in which case it is efficient to use Theorem 1, because there are at most 3 symbolic points to consider corresponding to the respective solutions of the equation. The quantified variable might only occur in inequalities, however. Consider a formula of the form

$$\exists x (ax^2 + bx + c \leq 0 \wedge F) \quad (x \notin a, b, c) \quad (2)$$

where  $x$  does not occur in  $a, b, c$ . Under the respective conditions from Theorem 1, the possible solutions  $-c/b, (-b + \sqrt{d})/(2a), (-b - \sqrt{d})/(2a)$  from (1) continue to be options for solutions of (2), because one way of satisfying the weak inequality  $ax^2 + bx + c \leq 0$  is by satisfying the equation  $ax^2 + bx + c = 0$ . So if  $F$  is *true* for any of those solutions of the quadratic equation (under the auspices of the additional constraints on  $a, b, c$ ), then (2) holds as well.

Yet, if those points do not work out, the weak inequality in (2) allows for more possible solutions than the equation does. For example, if  $a = 0, b > 0$ , then sufficiently small values of  $x$  would satisfy  $0x^2 + bx + c \leq 0$ . Also, if  $a < 0$ , then sufficiently small values of  $x$  would satisfy  $ax^2 + bx + c \leq 0$ , because  $x^2$  grows faster than  $x$  and, thus the negative  $ax^2$  ultimately overcomes any contribution of  $bx$  and  $c$  to the value of  $ax^2 + bx + c$ . But if we literally substituted each such smaller value of  $x$  into  $F$ , that would quickly diverge into the full substitution  $\bigvee_{t \in T} F_x^t$  for the un insightful case of  $T \stackrel{\text{def}}{=} \mathbb{R}$  from [Lecture 18](#).

Now, one possibility of pursuing this line of thought may be to substitute smaller and smaller values for  $x$  into (2) and see if one of those happens to work. There is a much better way though. The only really small value that would have to be substituted into (2) for  $x$  to see if it happens to work is one that is so negative that it is smaller than all others:  $-\infty$ , which is the lower limit of all negative real numbers. Alternatively,  $-\infty$  can be understood as being “always as negative as needed, i.e. more negative than anything else”. Think of  $-\infty$  as being built out of elastic rubber so that it always ends up being smaller when being compared to any actual real number, because the elastic number  $-\infty$  simply shrinks every time it is being compared to any other number. Analogously,  $\infty$  is the upper limit of all real numbers or “always as positive as needed, i.e. more positive than anything else”. The elastic rubber version of understanding  $\infty$  is such that  $\infty$  always grows as needed every time it is being compared to any other number.

Let  $\infty, -\infty$  be *positive and negative infinities*, respectively, i.e. choose extra elements  $\infty, -\infty \notin \mathbb{R}$  with  $-\infty < r < \infty$  for all  $r \in \mathbb{R}$ . Formulas of real arithmetic can be substituted with  $\pm\infty$  for a variable  $x$  in the compactified reals  $\mathbb{R} \cup \{\infty, -\infty\}$ . Yet, just like

with square root expressions,  $\pm\infty$  do not actually need to ever occur in the resulting formula, because substitution of infinities into formulas can be defined differently. For example,  $(x + 5 > 0)_x^\infty$  simplifies to *false*, while  $(x + 5 < 0)_x^\infty$  simplifies to *true*.

**Note 2.** Substitution of the infinity  $-\infty$  for  $x$  into an atomic formula for a polynomial  $p \stackrel{\text{def}}{=} \sum_{i=0}^n a_i x^i$  with polynomials  $a_i$  that do not contain  $x$  is defined by the following equivalences (accordingly for substituting  $\infty$  for  $x$ ).

$$(p = 0)_{\hat{x}}^{-\infty} \equiv \bigwedge_{i=0}^n a_i = 0 \quad (3)$$

$$(p \leq 0)_{\hat{x}}^{-\infty} \equiv (p < 0)_{\hat{x}}^{-\infty} \vee (p = 0)_{\hat{x}}^{-\infty} \quad (4)$$

$$(p < 0)_{\hat{x}}^{-\infty} \equiv p(-\infty) < 0 \quad (5)$$

$$(p \neq 0)_{\hat{x}}^{-\infty} \equiv \bigvee_{i=0}^n a_i \neq 0 \quad (6)$$

Lines (3) and its dual (6) use that the only equation of real arithmetic that infinities  $\pm\infty$  satisfy is the trivial equation  $0 = 0$ . Line (4) uses the equivalence  $p \leq 0 \equiv p < 0 \vee p = 0$  and is equal to  $(p < 0 \vee p = 0)_{\hat{x}}^{-\infty}$  by the substitution base from [Lecture 18](#). Line (5) uses a simple inductive definition based on the degree,  $\deg(p)$ , in the variable  $x$  of the polynomial  $p$  to characterize whether  $p$  is ultimately negative at  $-\infty$  (or for sufficiently negative numbers):

**Note 3.** Let  $p \stackrel{\text{def}}{=} \sum_{i=0}^n a_i x^i$  with polynomials  $a_i$  that do not contain  $x$ . Whether  $p$  is ultimately negative at  $-\infty$ , suggestively written  $p(-\infty) < 0$ , is easy to characterize by induction on the degree of the polynomial:

$$p(-\infty) < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \deg(p) = 0 \\ (-1)^n a_n < 0 \vee (a_n = 0 \wedge (\sum_{i=0}^{n-1} a_i x^i)(-\infty) < 0) & \text{if } \deg(p) > 0 \end{cases}$$

$p(-\infty) < 0$  is true in a state in which  $\lim_{x \rightarrow -\infty} p(x) < 0$ .

The first line captures that the sign of polynomials of degree 0 in the variable  $x$  does not depend on  $x$ , so  $p(-\infty) < 0$  iff the polynomial that has degree 0 in  $x$  and, thus, only consists of a term  $p = a_0$  that is constant in  $x$ , is negative (which may still depend on the value of other variables in  $a_0$  but not on  $x$ ). The second line captures that the sign at  $-\infty$  of a polynomial of degree  $n = \deg(p) > 0$  is determined by the degree-modulated sign of its leading coefficient  $a_n$ , because for  $x$  of sufficiently big value, the value of  $a_n x^n$  will dominate all lower-degree values, whatever their coefficients are. For even  $n > 0$ ,  $x^n < 0$  while  $x^n > 0$  for odd  $n$  at  $-\infty$ . In case the leading coefficient  $a_n$  evaluates to zero, the value of  $p$  at  $-\infty$  depends on the value at  $-\infty$  of the remaining polynomial  $\sum_{i=0}^{n-1} a_i x^i$  of lower degree, which can be determined recursively as  $(\sum_{i=0}^{n-1} a_i x^i)(-\infty) < 0$ .

Substitution of  $\infty$  for  $x$  into an atomic formula is defined similarly, except that the sign factor  $(-1)^n$  disappears, because  $x^n > 0$  at  $\infty$  whatever value  $n > 0$  has. Substitution of  $\infty$  or of  $-\infty$  for  $x$  into other first-order formulas is then defined on this basis as in [Lecture 18](#).

*Example 2* (Sign of quadratic polynomials at  $-\infty$ ). Using this principle to check under which circumstance the quadratic inequality from (2) evaluates to *true* yields the answer from our earlier ad-hoc analysis of what happens for sufficiently small values of  $x$ :

$$\begin{aligned} (ax^2 + bx + c \leq 0)_{\hat{x}}^{-\infty} &\equiv (-1)^2 a < 0 \vee a = 0 \wedge ((-1)b < 0 \vee b = 0 \wedge c < 0) \\ &\equiv a < 0 \vee a = 0 \wedge (b > 0 \vee b = 0 \wedge c < 0) \end{aligned}$$

One representative example for each of those cases is illustrated in Fig. 1. In the same way, the virtual substitution can be used to see under which circumstance the remainder formula  $F$  from (2) also evaluates to *true* for sufficiently small values of  $x$ , exactly when  $F_{\hat{x}}^{-\infty}$  holds. In contrast, note that (at least for  $a \neq 0$ ), the virtual substitution of  $\infty$  for  $x$  would not make sense to check (2) at, because in that case, the inequality  $ax^2 + bx + c \leq 0$  is violated, as can be confirmed by checking  $(ax^2 + bx + c \leq 0)_{\hat{x}}^{\infty}$ . That would be different if the inequality were  $ax^2 + bx + c \geq 0$ .

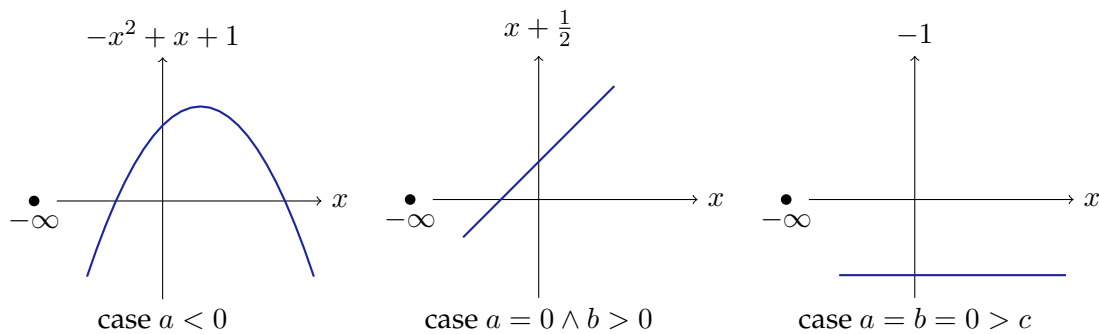


Figure 1: Illustration of the value of different quadratic functions  $p$  where  $p_{\hat{x}}^{-\infty} \equiv true$

The crucial thing to note is again that the *virtual substitution* of infinities  $\pm\infty$  for  $x$  in  $F$  giving  $F_{\hat{x}}^{\pm\infty}$  is semantically equivalent to the result  $F_x^{\pm\infty}$  of the literal substitution replacing  $x$  with  $\pm\infty$ , but operationally different, because the virtual substitution never introduces actual infinities. Because of their semantical equivalence, we use the same notation by abuse of notation.

**Lemma 3** (Virtual substitution lemma for infinities). *The result  $F_{\hat{x}}^{-\infty}$  of the virtual substitution is semantically equivalent to the the result  $F_x^{-\infty}$  of the literal substitution, but better behaved, because it stays within  $\text{FOL}_{\mathbb{R}}$  proper. Essentially, the following equivalence of virtual substitution and literal substitution for infinities is valid:*

$$F_x^{-\infty} \leftrightarrow F_{\hat{x}}^{-\infty}$$

*Keep in mind that the result  $F_{\hat{x}}^{-\infty}$  of virtual substitution is a proper formula of  $\text{FOL}_{\mathbb{R}}$ , while the literal substitution  $F_x^{-\infty}$  could only be considered a formula in an extended logic such as  $\text{FOL}_{\mathbb{R} \cup \{-\infty, \infty\}}$  that allows for infinite quantities. The same property holds for  $F_{\hat{x}}^{\infty}$ .*

Note that the situation is, in a sense, the converse of [Lecture 18](#), where the square root expressions were already in the semantic domain  $\mathbb{R}$ , and just had to be made accessible in the syntactic formulas via virtual substitutions. In [Lemma 3](#), instead, virtual substitutions already know more about infinities  $\pm\infty$  than the semantic domain  $\mathbb{R}$  does, which is why the semantic domain needs an extension to  $\mathbb{R} \cup \{-\infty, \infty\}$  for the alignment in [Lemma 3](#).

**Expedition 1** (Infinite challenges with infinities in extended reals  $\mathbb{R} \cup \{-\infty, \infty\}$ ). The set  $\mathbb{R} \cup \{-\infty, \infty\}$  is seemingly easily written down as a semantic domain of extended reals. What exactly do we mean by it? The set of reals to which we adjoin two new elements, denoted  $-\infty$  and  $\infty$  which are the minimum and maximum element of the ordering  $\leq$ :

$$\forall x (-\infty \leq x \leq \infty) \tag{7}$$

This turns  $\mathbb{R} \cup \{-\infty, \infty\}$  into a complete lattice, because every subset has a supremum and an infimum. The extended reals are a compactification of  $\mathbb{R}$ . But where does that leave the other arithmetic properties of  $\mathbb{R}$ ? What is  $\infty + 1$  or  $\infty + x$  when  $\infty$  is already infinitely big? The compatibility of  $\leq$  with  $+$  expects  $\infty \leq \infty + x$  at least for all  $x \geq 0$ . By (7) also  $\infty + x \leq \infty$ . Because  $\infty$  is so infinitely big, the same  $\infty + x = \infty$  is expected even for all  $x$ , except  $-\infty$ . The compatibility of  $\leq$  with  $\cdot$  expects  $\infty \leq \infty \cdot x$  at least for all  $x \geq 1$ . By (7) also  $\infty \cdot x \leq \infty$ . Since  $\infty$  is infinitely big, the same  $\infty \cdot x = \infty$  is expected even for all  $x > 0$ .

$$\begin{array}{ll} \infty + x = \infty & \text{for all } x \neq -\infty \\ -\infty + x = -\infty & \text{for all } x \neq \infty \\ \infty \cdot x = \infty & \text{for all } x > 0 \\ \infty \cdot x = -\infty & \text{for all } x < 0 \\ -\infty \cdot x = -\infty & \text{for all } x > 0 \\ -\infty \cdot x = \infty & \text{for all } x < 0 \end{array}$$

This extension sounds reasonable. But the resulting set  $\mathbb{R} \cup \{-\infty, \infty\}$  is not a field. Otherwise  $\infty$  would have an additive inverse. But what  $x$  would satisfy  $\infty + x = 0$ ? One might guess  $x = -\infty$ , but then one would also expect  $0 = \infty + (-\infty) = \infty + (-\infty + 1) = ((\infty + (-\infty)) + 1) = 0 + 1 = 1$ , which is not a good idea to adopt for proving anything in a sound way. Instead, problematic terms remain explicitly undefined.

$$\begin{aligned}\infty - \infty &= \text{undefined} \\ 0 \cdot \infty &= \text{undefined} \\ \pm\infty / \pm\infty &= \text{undefined} \\ 1/0 &= \text{undefined}\end{aligned}$$

Since these conventions make infinities somewhat subtle, we happily remember that the only thing we need them for is to make sense of inserting sufficiently negative (or sufficiently positive) numbers into inequalities to satisfy them. That is still mostly harmless.

## 4. Infinitesimal $\varepsilon$ Substitutions

Theorem 1 addresses the case where the quantified variable occurs in a linear or quadratic equation and the virtual substitution in Sect. 3 adds the case of sufficiently small values for  $x$  to handle  $ax^2 + bx + c \leq 0$ . Consider a formula of the form

$$\exists x (ax^2 + bx + c < 0 \wedge F) \quad (x \notin a, b, c) \quad (8)$$

In this case, the roots from Theorem 1 will not help, because they satisfy the equation  $ax^2 + bx + c = 0$  but not the strict inequality  $ax^2 + bx + c < 0$ . The virtual substitution of  $-\infty$  for  $x$  from Sect. 3 still makes sense to consider, because the arbitrarily small negative numbers that it corresponds to might satisfy  $F$  and  $ax^2 + bx + c < 0$ . If  $-\infty$  does not work, however, the solution of (8) could be near one of the roots of  $ax^2 + bx + c = 0$ , just slightly off so that  $ax^2 + bx + c < 0$  is actually satisfied rather than  $ax^2 + bx + c = 0$ . How far off? Well, saying that exactly by any real number is again difficult, because any particular real number might already have been too large in absolute value, depending on the constraints in the remainder of  $F$ . Again, this calls for quantities that are always as small as we need them to be.

Sect. 3 used a negative quantity that is so small that it is smaller than all negative numbers and hence infinitely small (but infinitely large in absolute value). The negative infinity  $-\infty$  that is smaller no matter what other number we compare it with. Analyzing (8) needs *positive* quantities that are infinitely small and hence also infinitely small in absolute value. Infinitesimals are positive quantities that are always smaller than all positive real numbers, i.e. “always as small as needed”. Think of them as built out of elastic rubber so that they always shrink as needed when compared with any actual positive real number so that the infinitesimals end up being smaller than posi-

tive reals. Another way of looking at infinitesimals is that they are the multiplicative inverses of  $\pm\infty$ .

A positive infinitesimal  $\infty > \varepsilon > 0$  is positive and an extended real that is infinitesimal, i.e. positive but smaller than all positive real numbers ( $\varepsilon < r$  for all  $r \in \mathbb{R}$  with  $r > 0$ ). For all non-zero polynomials  $p \in \mathbb{R}[x] \setminus \{0\}$ ,  $\zeta \in \mathbb{R}$ , the Taylor series

$$p(\zeta + \varepsilon) = \sum_{n=0}^{\infty} \frac{p^{(n)}(\zeta)}{n!} (\zeta + \varepsilon - \zeta)^n = \sum_{n=0}^{\infty} \frac{p^{(n)}(\zeta)}{n!} \varepsilon^n = \sum_{n=0}^{\deg(p)} \frac{p^{(n)}(\zeta)}{n!} \varepsilon^n$$

of  $p$  around  $\zeta$  evaluated at  $\zeta + \varepsilon$  (note that  $\varepsilon$  is small enough to be in the domain of convergence of the Taylor series) can be used to show:

1.  $p(\zeta + \varepsilon) \neq 0$   
that is, infinitesimals  $\varepsilon$  are always so small that they never yield roots of any equation, except the trivial zero polynomial. Whenever it looks like there might be a root, the infinitesimal just became a bit smaller to avoid satisfying the equation. And nonzero univariate polynomials  $p(x)$  only have finitely many roots, so the infinitesimals will take care to avoid all of them by becoming just a little smaller.
2.  $p(\zeta) \neq 0 \Rightarrow p(\zeta)p(\zeta + \varepsilon) > 0$ ,  
that is,  $p$  has constant sign on infinitesimal neighborhoods of nonroots  $\zeta$ . If the neighborhood around  $\zeta$  is small enough (and for an infinitesimal it will be), then the polynomial will not yet have changed sign on that interval, because the sign will only change after passing one of the roots.
3.  $0 = p(\zeta) = p'(\zeta) = p''(\zeta) = \dots = p^{(k-1)}(\zeta) \neq p^{(k)}(\zeta) \Rightarrow p^{(k)}(\zeta)p(\zeta + \varepsilon) > 0$ ,  
that is the first nonzero derivative of  $p$  at  $\zeta$  determines the sign of  $p$  in a small enough neighborhoods of  $\zeta$  (infinitesimal neighborhoods will be small enough), because the sign will only change after passing one of the roots.

**Note 5.** Substitution of an infinitesimal expression  $e + \varepsilon$  with a square root expression  $e = (a + b\sqrt{c})/d$  and a positive infinitesimal  $\varepsilon$  for  $x$  into a polynomial  $p = \sum_{i=0}^n a_i x^i$  with polynomials  $a_i$  that do not contain  $x$  is defined by the following equivalences.

$$(p = 0)_{\hat{x}}^{e+\varepsilon} \equiv \bigwedge_{i=0}^n a_i = 0 \quad (9)$$

$$(p \leq 0)_{\hat{x}}^{e+\varepsilon} \equiv (p < 0)_{\hat{x}}^{e+\varepsilon} \vee (p = 0)_{\hat{x}}^{e+\varepsilon} \quad (10)$$

$$(p < 0)_{\hat{x}}^{e+\varepsilon} \equiv (p^+ < 0)_{\hat{x}}^e \quad (11)$$

$$(p \neq 0)_{\hat{x}}^{e+\varepsilon} \equiv \bigvee_{i=0}^n a_i \neq 0 \quad (12)$$

Lines (9) and its dual (12) use that infinitesimals offsets satisfy no equation except the trivial equation  $0=0$  (case 1), which makes infinitesimals and infinities behave the same



as far as equations go. Line (10) again uses the equivalence  $p \leq 0 \equiv p < 0 \vee p = 0$ . Line (11) checks whether the sign of  $p$  at the square root expression  $e$  is already negative (which will make  $p$  inherit the same negative sign after an infinitesimal offset at  $e + \varepsilon$  by case 2) or will immediately become negative right away using a recursive formulation of immediately becoming negative that uses higher derivatives (which determine the sign by case 3). The lifting to arbitrary quantifier-free formulas of real arithmetic is again by substitution into all atomic subformulas and equivalences such as  $(p > q) \equiv (p - q > 0)$  as defined in Lecture 18. Note that, for the case  $(p < 0)_{\dot{x}}^{e+\varepsilon}$ , the (non-infinitesimal) square root expression  $e$  gets virtually substituted in for  $x$  into a formula  $p^+ < 0$ , which characterizes whether  $p$  becomes negative immediately at or after  $x$  (which will be virtually substituted by the intended square root expression  $e$  momentarily).

**Note 6.** Whether  $p$  is immediately negative at  $x$ , i.e. negative itself or of a derivative  $p'$  that makes it negative on an infinitesimal interval  $[x, x + \varepsilon]$ , suggestively written  $p^+ < 0$ , can be characterized recursively:

$$p^+ < 0 \stackrel{\text{def}}{\equiv} \begin{cases} p < 0 & \text{if } \text{deg}(p) = 0 \\ p < 0 \vee (p = 0 \wedge (p')^+ < 0) & \text{if } \text{deg}(p) > 0 \end{cases}$$

$p^+ < 0$  is true in a state in which  $\lim_{y \rightarrow x^+} p(x) = \lim_{y \searrow x} p(x) = \lim_{\substack{y > x \\ y \rightarrow x}} p(x) < 0$  holds for the limit of  $p$  at  $x$  from the right.

The first line captures that the sign of polynomials of degree 0 in the variable  $x$  does not depend on  $x$ , so they are negative at  $x$  iff the polynomial  $p = a_0$  that has degree 0 in  $x$  is negative (which may still depend on the value of other variables in  $a_0$ ). The second line captures that the sign at  $x + \varepsilon$  of a non-constant polynomial is still negative if it is negative at  $x$  (because  $x + \varepsilon$  is not far enough away from  $x$  for any sign changes by case 2) or if  $x$  is a root of  $p$  but its derivative  $p'$  at  $x$  is immediately negative, since the first nonzero derivative at  $x$  determines the sign near  $x$  by case 3.

*Example 4* (Sign of quadratic polynomials after second root). Using this principle to check under which circumstance the quadratic strict inequality from (8) evaluates to true at  $(-b + \sqrt{b^2 - 4ac})/(2a) + \varepsilon$ , i.e. right after its root  $(-b + \sqrt{b^2 - 4ac})/(2a)$ , leads to the following computation.

$$(ax^2 + bx + c)^+ < 0 \equiv ax^2 + bx + c < 0 \vee ax^2 + bx + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0)$$

with successive derivatives to break ties (i.e. zero signs in previous derivatives). Hence,

$$\begin{aligned}
 (ax^2 + bx + c < 0)_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} &\equiv ((ax^2 + bx + c)^+ < 0)_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} \equiv \\
 (ax^2 + bx + c < 0 \vee ax^2 + bc + c = 0 \wedge (2ax + b < 0 \vee 2ax + b = 0 \wedge 2a < 0))_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} &\equiv \\
 \equiv 0 \cdot 1 < 0 \vee 0 = 0 \wedge \underbrace{(0 < 0 \vee 4a^2 \leq 0 \wedge (0 < 0 \vee -4a^2(b^2 - 4ac) < 0))}_{(2ax+b<0)_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)}} \vee \underbrace{0 = 0}_{(2ax+b=0)_{\hat{x}}} \wedge \underbrace{2a1 < 0}_{2a<0} &\equiv \\
 &\equiv 4a^2 \leq 0 \wedge -4a^2(b^2 - 4ac) < 0 \vee 2a < 0
 \end{aligned}$$

because the square root virtual substitution of its own root  $(-b + \sqrt{b^2 - 4ac})/(2a)$  into  $ax^2 + bx + c$  gives  $(ax^2 + bx + c)_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)} = 0$  by construction (compare example from [Lecture 18](#)). The virtual substitution into the other polynomial  $2ax + b$  above computes as follows:

$$\begin{aligned}
 (2ax + b)_{\hat{x}}^{(-b \pm \sqrt{b^2 - 4ac})/(2a)} &\equiv 2a \cdot (-b \pm \sqrt{b^2 - 4ac})/(2a) + b \\
 &= (-2ab + \pm 2a\sqrt{b^2 - 4ac})/(2a) + b \\
 &= (\cancel{-2ab} + \cancel{2ab} + \pm 2a\sqrt{b^2 - 4ac})/(2a) \\
 &= (0 + \pm 2a\sqrt{b^2 - 4ac})/(2a)
 \end{aligned}$$

The resulting formula can be further simplified internally to

$$(ax^2 + bx + c < 0)_{\hat{x}}^{(-b+\sqrt{b^2-4ac})/(2a)+\varepsilon} \equiv 4a^2 \leq 0 \wedge -4a^2(b^2 - 4ac) < 0 \vee 2a < 0 \equiv 2a < 0$$

because the first conjunct  $4a^2 \leq 0 \equiv a = 0$  and, with  $a = 0$ , the second conjunct simplifies to  $-4a^2(b^2 - 4ac)_a^0 = -0(b^2) < 0$ , which is impossible in the reals. This answer makes sense. Because, indeed, exactly if  $2a < 0$  will a quadratic polynomial still evaluate to  $ax^2 + bx + c < 0$  right after its second root  $(-b + \sqrt{b^2 - 4ac})/(2a)$ . Fig. 2 illustrates to how this relates to the parabola point downwards, because of  $2a < 0$ .

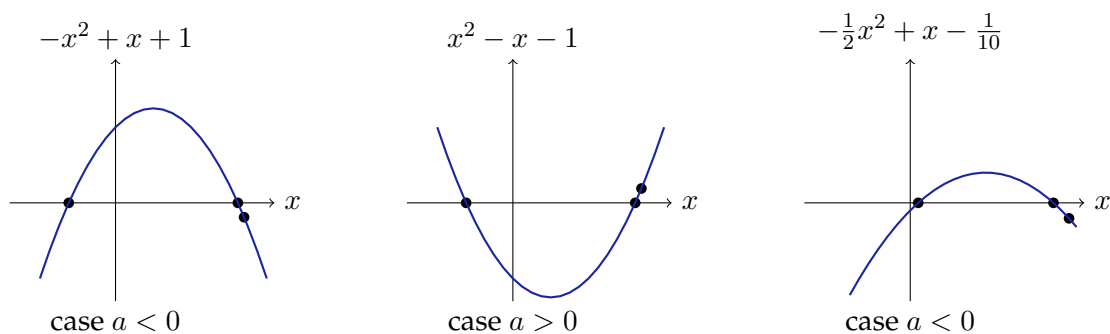


Figure 2: Illustration of the sign after the second root for quadratic functions  $p$

Formulas such as this one ( $2a > 0$ ) are the result of a quantifier elimination procedure. If the formula after quantifier elimination is either *true* or *false*, then you know for sure that the formula is valid (*true*) or unsatisfiable (*false*), respectively. If the result of quantifier elimination is *true*, for example, KeYmaera can close proof branches (marked by proof rule  $\mathbb{R}$  in our sequent proofs). Yet, quantifier elimination can also return other formulas, such as  $2a > 0$ , which are equivalent to the formula where quantifier elimination has been applied. In particular, they identify exactly under which circumstance that respective quantified formula is true. This can be very useful for identifying the missing assumptions to make a proof work and the corresponding statement true.

**Note 7** (Quantifier elimination identifies requirements). *If the result of quantifier elimination is true, the corresponding formula is valid. If it is false, the corresponding formula is not valid (and even unsatisfiable). In between, i.e. when quantifier elimination results in a logical formula that is sometimes false and sometimes true, then this formula identifies exactly the missing requirements that are needed to make the desired formula true. This can be useful to synthesize missing requirements. Take care, however, not to work with universal closures, in which case true and false are the only possible outcomes.*

The crucial thing to note about the process is again that the *virtual substitution* of infinitesimal expressions  $e + \varepsilon$  for  $x$  in  $F$  giving  $F_{\hat{x}}^{e+\varepsilon}$  is semantically equivalent to the result  $F_x^{e+\varepsilon}$  of the literal substitution replacing  $x$  with  $e + \varepsilon$ , but operationally different, because the virtual substitution never introduces actual infinitesimals. Because of their semantical equivalence, we use the same notation by abuse of notation.

**Lemma 5** (Virtual substitution lemma for infinitesimals). *The result  $F_{\hat{x}}^{e+\varepsilon}$  of the virtual substitution is semantically equivalent to the the result  $F_x^{e+\varepsilon}$  of the literal substitution, but better behaved, because it stays within  $\text{FOL}_{\mathbb{R}}$  proper. Essentially, the following equivalence of virtual substitution and literal substitution for infinitesimals is valid:*

$$F_x^{e+\varepsilon} \leftrightarrow F_{\hat{x}}^{e+\varepsilon}$$

*Keep in mind that the result  $F_{\hat{x}}^{e+\varepsilon}$  of virtual substitution is a proper formula of  $\text{FOL}_{\mathbb{R}}$ , while the literal substitution  $F_x^{e+\varepsilon}$  could only be considered a formula in an extended logic such as  $\text{FOL}_{\mathbb{R}[\varepsilon]}$  that allows for infinitesimal quantities from nonstandard analysis.*

Computationally more efficient substitutions of infinitesimals have been reported elsewhere [BD07].

**Expedition 2** (Nonstandard analysis: infinite challenges with infinitesimal  $\varepsilon$ ). Infinite quantities in the extended reals  $\mathbb{R} \cup \{-\infty, \infty\}$  already needed some attention to stay away from undefined expressions. Infinitesimals are infinitely more subtle than infinities. Real numbers are Archimedean, i.e. for every non-zero  $x \in \mathbb{R}$ , there

is an  $n \in \mathbb{N}$  such that

$$\underbrace{|x + x + \cdots + x|}_{n \text{ times}} > 1$$

Infinitesimals are non-Archimedean, because it does not matter how often you add  $\varepsilon$ , it still won't sum to one. There is a myriad of ways of making sense of infinitesimal quantities in nonstandard analysis, including surreal numbers, superreal numbers, and hyperreals. In a sense, infinitesimal quantities can be considered as multiplicative inverses of infinities but bring up many subtleties. For example, if an infinitesimal  $\varepsilon$  is added to  $\mathbb{R}$ , then the following terms need to denote values and satisfy ordering relations:

$$\varepsilon^2 \quad \varepsilon \quad x^2 + \varepsilon \quad (x + \varepsilon)^2 \quad x^2 + 2\varepsilon x + 5\varepsilon + \varepsilon^2$$

Fortunately, a rather tame version of infinitesimals is enough for the context of virtual substitution. The crucial properties of infinitesimals we need are [dMP13]:

$$\begin{aligned} \varepsilon &> 0 \\ \forall x \in \mathbb{R} (x > 0 &\rightarrow \varepsilon < x) \end{aligned}$$

That is, the infinitesimal  $\varepsilon$  is positive and smaller than all positive reals.

## 5. Quantifier Elimination by Virtual Substitution

The following quantifier elimination technique due to Weispfenning [Wei97] works for formulas with a quantified variable that occurs at most quadratically.

**Theorem 6** (Virtual substitution of quadratic constraints [Wei97]). *Let  $F$  be a quantifier-free formula in which all atomic formulas are of the form  $ax^2 + bx + c \sim 0$  for  $x$ -free polynomials  $a, b, c$  (i.e.  $x \notin a, b, c$ ) and  $\sim \in \{=, \leq, <, \neq\}$ , with corresponding discriminant  $d \stackrel{\text{def}}{=} b^2 - 4ac$ . Then  $\exists x F$  is equivalent over  $\mathbb{R}$  to the following quantifier-free formula:*

$$\begin{aligned} &F_{\hat{x}}^{-\infty} \\ \vee \bigvee_{(ax^2+bx+c \{ \leq \} 0) \in F} &(a = 0 \wedge b \neq 0 \wedge F_{\hat{x}}^{-c/b} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\hat{x}}^{(-b+\sqrt{d})/(2a)} \vee F_{\hat{x}}^{(-b-\sqrt{d})/(2a)})) \\ \vee \bigvee_{(ax^2+bx+c \{ < \} 0) \in F} &(a = 0 \wedge b \neq 0 \wedge F_{\hat{x}}^{-c/b+\varepsilon} \vee a \neq 0 \wedge d \geq 0 \wedge (F_{\hat{x}}^{(-b+\sqrt{d})/(2a)+\varepsilon} \vee F_{\hat{x}}^{(-b-\sqrt{d})/(2a)+\varepsilon})) \end{aligned}$$

*Proof.* The proof is an extended form of the proof reported in the literature [Wei97]. The proof first considers the literal substitution of square root expressions, infinities, and infinitesimals and then, as a second step, uses that the virtual substitutions that avoid square root expressions, infinities, and infinitesimals are equivalent (Lecture 18, Lemma 3 and 5). Let  $G$  denote the quantifier-free right-hand side so that the validity of

the following formula needs to be shown:

$$\exists x F \leftrightarrow G \tag{13}$$

The implication from the quantifier-free formula  $G$  to  $\exists x F$  in (13) is obvious, because each disjunct of the quantifier-free formula has a conjunct of the form  $F_x^t$  for some (extended) term  $t$  even if it may be a square root expression or infinity or term involving infinitesimals. Whenever a formula of the form  $F_x^t$  is true,  $\exists x F$  holds with that  $t$  as a witness, even when  $t$  is a square root expression, infinity, or infinitesimal.

The converse implication from  $\exists x F$  to the quantifier-free formula  $G$  in (13) depends on showing that the quantifier-free formula  $G$  covers all possible representative cases and that the accompanying constraints on  $a, b, c, d$  are necessary so that they do not constrain solutions in unjustified ways.

One key insight is that it is enough to prove (13) for the case where all variables in  $F$  except  $x$  have concrete numeric real values, because the equivalence (13) is valid iff it true in all states. So considering one concrete state at a time is enough. By a fundamental property of real arithmetic called *o-minimality*, the set

$$\mathcal{S}(F) = \{\nu(x) \in \mathbb{R} : \nu \models F\}$$

of all real values for  $x$  that satisfy  $F$  forms a finite union of (pairwise disjoint) intervals, because the polynomials in  $F$  only change signs at their roots. There are only finitely many roots now that the polynomials have become univariate, i.e. with the only variable  $x$ , since all free variables are evaluated to concrete real numbers in  $\nu$ . Without loss of generality (by merging overlapping or adjacent intervals), all those intervals are assume to be maximal, i.e. no bigger interval would satisfy  $F$ . So  $F$  actually changes its truth-value at the lower and upper endpoints of these intervals (unless the interval is unbounded).

The endpoints of these intervals are of the form  $-c/b$ ,  $(-b + \sqrt{d})/(2a)$ ,  $(-b - \sqrt{d})/(2a)$  or  $\infty$ ,  $-\infty$  for any of the polynomials  $ax^2 + bx + c$  in  $F$ , because all polynomials in  $F$  are at most quadratic and all roots of those polynomials are of one of the above forms. In particular, if  $-c/b$  is an endpoint of the intervals of  $\mathcal{S}(F)$  for a polynomial  $ax^2 + bx + c$  in  $F$ , then  $a = 0$ ,  $b \neq 0$ , because that is the only case where  $-c/b$  satisfies  $F$ , which has only at most quadratic polynomials. Likewise, if  $(-b + \sqrt{d})/(2a)$  or  $(-b - \sqrt{d})/(2a)$  are endpoints of intervals of  $\mathcal{S}(F)$  for a polynomial  $ax^2 + bx + c$  in  $F$ , then both imply that  $a \neq 0$  and discriminant  $d \geq 0$ , otherwise there is no such solution in the reals. Consequently, all the side conditions for the roots in the quantifier-free formula  $G$  are necessary.

Now consider one interval  $I \subseteq \mathcal{S}(F)$  (if there is none,  $\exists x F$  is *false* and so will  $G$  be). If  $I$  has no lower bound in  $\mathbb{R}$ , then  $F_{\hat{x}}^{-\infty}$  is true by construction (by Lemma 3, the virtual substitution  $F_{\hat{x}}^{-\infty}$  is equivalent to the literal substitution  $F_x^{-\infty}$  in  $\pm\infty$ -extended real arithmetic). Otherwise, let  $\alpha \in \mathbb{R}$  be the lower bound of  $I$ . If  $\alpha \in I$  (i.e.  $I$  is closed at the lower bound), then  $\alpha$  is of the form  $-c/b$ ,  $(-b + \sqrt{d})/(2a)$ ,  $(-b - \sqrt{d})/(2a)$  for some equation  $(ax^2 + bx + c = 0) \in F$  or some weak inequality  $(ax^2 + bx + c \leq 0) \in F$

from  $F$ . Since the respective extra conditions on  $a, b, c, d$  hold, the quantifier-free formula  $G$  evaluates to true. If, otherwise,  $\alpha \notin I$  (i.e.  $I$  is open at the lower bound  $\alpha$ ), then  $\alpha$  is of the form  $-c/b, (-b + \sqrt{d})/(2a), (-b - \sqrt{d})/(2a)$  for some disequation  $(ax^2 + bx + c \neq 0) \in F$  or some strict inequality  $(ax^2 + bx + c < 0) \in F$ . Hence, the interval  $I$  cannot be a single point. Thus, one of the infinitesimal increments  $-c/b + \varepsilon, (-b + \sqrt{d})/(2a) + \varepsilon, (-b - \sqrt{d})/(2a) + \varepsilon$  is in  $I \subseteq \mathcal{S}(F)$ , because infinitesimals are smaller than all positive real numbers. Since the respective conditions  $a, b, c, d$  hold, the quantifier-free formula  $G$  is again true. Hence, in either case, the quantifier-free formula is equivalent to  $\exists x F$  in state  $\nu$ . Since the state  $\nu$  giving concrete real numbers to all free variables of  $\exists x F$  was arbitrary, the same equivalence holds for all  $\nu$ , which means that the quantifier-free formula  $G$  is equivalent to  $\exists x F$ . That is  $G \leftrightarrow \exists x F$  is valid, i.e.  $\models G \leftrightarrow \exists x F$ .  $\square$

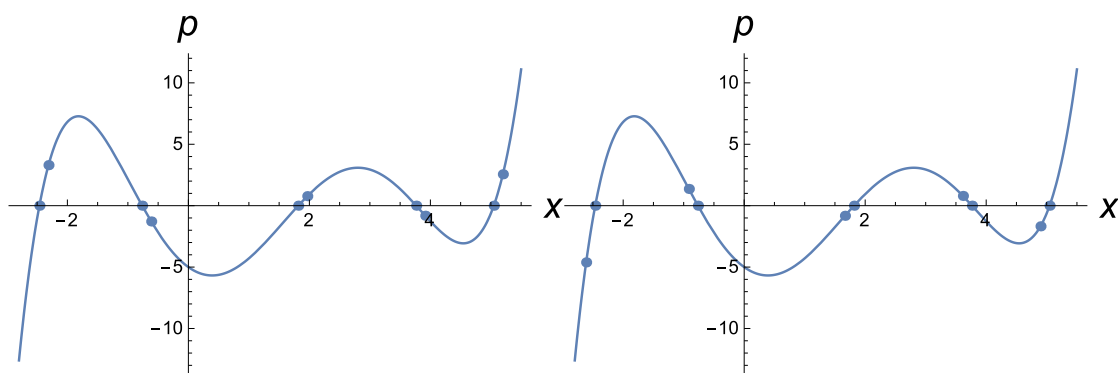


Figure 3: **(left)** Illustration of roots  $e$  and infinitesimal offsets  $e + \varepsilon$  checked by virtual substitution along with  $-\infty$  **(right)** Illustration of roots  $e$  and infinitesimal offsets  $e - \varepsilon$  that could be checked along with  $+\infty$  instead

The order of the interval endpoints that the proof of Theorem 6 uses in addition to  $-\infty$  are illustrated in Fig. 3(left). Observe that exactly one representative point is placed in each of the regions of interest,  $-\infty$ , each of the roots  $r$ , and just infinitesimally after the roots at  $e + \varepsilon$ . Alternatively, Theorem 6 could be rephrased to work with  $\infty$ , at each root  $e$ , and always before the roots at  $e - \varepsilon$ ; see Fig. 3(right) and Exercise 3. The illustrations in Fig. 3 show the ordering situation for a higher-degree polynomial  $p$  even if Theorem 6 only makes use of the argument for  $p = ax^2 + bx + c$  up to degree 2. Quantifier elimination procedures for higher degrees are still based on this fundamental principle, but require more algebraic computations.

Finally note that it is quite possible that the considered polynomial  $p$  does not single out the appropriate root  $e$  or off-root  $e + \varepsilon$  that satisfies  $F$  to witness  $\exists x F$ . So none of the points illustrated in Fig. 3 will satisfy  $F$ , because only a point other than  $e + \varepsilon$  in the open interval between two roots will work.

**Note 10** (No rejection without mention). *The key argument underlying all quantifier elimination procedures in some way or another is that another part of  $F$  that is not satisfied for any of the points in Fig. 3 that  $p$  brings about would have to mention another polynomial  $q$  with different roots  $d$  or different off-roots  $d + \varepsilon$  that will then enter the big disjunction in Theorem 6.*

*Example 7.* The example of nonnegative roots of quadratic polynomials from [Lecture 18 on Virtual Substitutions & Real Equations](#) used Theorem 1 to construct and justify the quantifier elimination equivalence:

$$\begin{aligned} \text{QE}(\exists x (ax^2 + bx + c = 0 \wedge x \geq 0)) \\ \equiv b^2 - 4ac \geq 0 \wedge (ba \leq 0 \wedge ac \geq 0 \vee a \geq 0 \wedge ac \leq 0 \vee a \leq 0 \wedge ac \leq 0) \end{aligned}$$

under the assumption  $a \neq 0$ . Specializing to a case shown in Fig. 2 gives:

$$\text{QE}(\exists x (x^2 - x + c = 0 \wedge x \geq 0)) \equiv (-1)^2 - 4c \geq 0 \wedge (c \geq 0 \vee c \leq 0) \equiv 1 - 4c \geq 0 \equiv c \leq \frac{1}{4}$$

By Theorem 6, the same square root expression substitution as in [Lecture 18 on Virtual Substitutions & Real Equations](#) will happen for the atomic formula  $x^2 - x + c \leq 0$  except that the case of  $-\infty$  will be added as well as the root 0 that results from considering the linear atomic formula  $-x \geq 0$ :

$$\begin{aligned} \text{QE}(\exists x (x^2 - x + c \leq 0 \wedge x \geq 0)) \equiv \\ \underbrace{(x^2 - x + c \leq 0 \wedge \dots)_{\hat{x}}^{-\infty}}_{\text{false}} \vee 1 - 4c \geq 0 \vee \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_{\hat{x}}^0}_{c \leq 0 \wedge 0 \geq 0} \equiv 1 - 4c \geq 0 \end{aligned}$$

Note that the additional disjunction  $x \leq 0$  coming from the root 0 of  $-x$  is in this case subsumed by the previous disjunct  $1 - 4c \geq 0$ . Hence, adding the roots of  $-x$  did not modify the answer in this case. When adding a third conjunct  $-x + 2 = 0$ , this becomes critical:

$$\text{QE}(\exists x (x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0))$$

Since the first two polynomials  $x^2 - x + c$  and  $-x$  are still the same, the same virtual substitutions will happen as before. Except that they fail on the new conjunct  $-x + 2 = 0$ , because the root 0 of the polynomial  $-x$  from the second conjunct does not satisfy  $-x + 2 = 0$  and because the virtual substitution of the roots  $(-1 \pm \sqrt{1 - 4c})/2$  of the first polynomial  $x^2 - x + c$  fail:

$$\begin{aligned} (-x + 2 = 0)_{\hat{x}}^{(-1 \pm \sqrt{1 - 4c})/2} \equiv ((1 + \mp 1\sqrt{1 - 4c})/2 + 2 = 0) \equiv ((3 + \mp 1\sqrt{1 - 4c})/2 = 0) \\ \equiv \mp 3 \leq 0 \wedge 3^2 - (\mp 1)^2(1 - 4c) = 0 \equiv -3 \leq 0 \wedge 3^2 - (-1)^2(1 - 4c) = 0 \equiv 8 - 4c = 0 \end{aligned}$$

The latter is only possible for  $c = 2$ , which is ruled out by the discriminant condition  $1 - 4c \geq 0$  that precedes it. And, indeed, neither the roots of the quadratic polynomial

illustrated in Fig. 2 nor the roots of  $-x$  nor  $-\infty$  are the right points to consider to satisfy the last conjunct. Of course, the last conjunct expresses that by saying  $-x + 2 = 0$  quite explicitly. Never mind that this is an equation for now. Either way, the atomic formula clearly exposes that  $-x + 2$  is the polynomial that it cares about. So its roots might be of interest and will, indeed, be considered in the big disjunction of Theorem 6 as well. Since  $-x + 2$  is a visibly linear polynomial, its solution is  $x = -2 / -1 = 2$  which is even kind enough to be a standard real number so that literal substitution is sufficient and no virtual substitution is needed. Consequently, the substitution of this root  $x = 2$  of the last conjunct into the full formula quickly yields:

$$(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)_x^2 \equiv 2^2 - 2 + c \leq 0 \wedge 2 \geq 0 \wedge 0 = 0 \equiv 2 + c \leq 0$$

This provides an answer that the quadratic polynomial  $x^2 - x + c$  itself could not foresee because it depends on the polynomial  $-x + 2$  to even take this root into consideration. By Theorem 6, the overall result of quantifier elimination, thus, is the combination of the cases considered separately above:

$$\begin{aligned} & \text{QE}(\exists x (x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)) \\ & \equiv \underbrace{(x^2 - x + c \leq 0 \wedge \dots)_x^{-\infty}}_{\text{false}} \\ & \vee 1 - 4c \geq 0 \wedge \underbrace{(\dots \wedge -x + 2 = 0)_x^{(-1 \pm \sqrt{1-4c})/2}}_{8-4c=0} \\ & \vee -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0)_x^0}_{c \leq 0 \wedge 0 \geq 0} \wedge \underbrace{(-x + 2 = 0)_x^0}_{2=0} \\ & \vee -1 \neq 0 \wedge \underbrace{(x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 = 0)_x^2}_{2+c \leq 0} \\ & \equiv 2 + c \leq 0 \equiv c \leq -2 \end{aligned}$$

In this particular case, observe that Theorem 1 using  $-x + 2 = 0$  as the key formula would have been most efficient, because that would have gotten the answer right away without fruitless disjunctions. This illustrates that it pays off to pay attention with real arithmetic and always choose the computationally most parsimonious approach. But the example also illustrates that the same computation would happen if the third conjunct would have been  $-x + 2 \leq 0$ , in which case Theorem 1 would not have helped.

Optimizations are possible for virtual substitution [Wei97] if there is only one quadratic occurrence of  $x$ , and that occurrence is not in an equation. If that occurrence is an equation, Theorem 1 already showed what to do. If there is only one occurrence of a quadratic inequality, the following variation of Theorem 6 works, which uses exclusively linear fractions.



**Note 11** ([Wei97]). Let  $\left( Ax^2 + Bx + C \left\{ \begin{array}{l} \leq \\ < \\ \neq \end{array} \right\} 0 \right) \in F$  be the only quadratic occurrence of  $x$ . In that case,  $\exists x F$  is equivalent over  $\mathbb{R}$  to the following quantifier-free formula:

$$\begin{aligned} & A = 0 \wedge B \neq 0 \wedge F_{\hat{x}}^{-C/B} \vee A \neq 0 \wedge F_{\hat{x}}^{-B/(2A)} \\ & \vee F_{\hat{x}}^{-\infty} \vee F_{\hat{x}}^{\infty} \\ & \vee \bigvee_{(0x^2+bx+c \left\{ \begin{array}{l} = \\ \leq \end{array} \right\} 0) \in F} (b \neq 0 \wedge F_{\hat{x}}^{-c/b}) \\ & \vee \bigvee_{(0x^2+bx+c \left\{ \begin{array}{l} \neq \\ < \end{array} \right\} 0) \in F} (b \neq 0 \wedge (F_{\hat{x}}^{-c/b+\varepsilon} \vee F_{\hat{x}}^{-c/b-\varepsilon})) \end{aligned}$$

The clou in this case is that the extremal values of  $Ax^2 + Bx + C$  are at the roots of the derivative

$$(Ax^2 + Bx + C)' = 2AX + B \stackrel{!}{=} 0 \text{ i.e. } x = -\frac{B}{2A}$$

Since the only quadratic occurrence in Note 11 is not an equation, this extremal value is the only point of the quadratic polynomial that matters. In this case,  $F_{\hat{x}}^{-B/(2A)}$  will substitute  $-B/(2A)$  for  $x$  in the only quadratic polynomial as follows:

$$\left( Ax^2 + Bx + C \left\{ \begin{array}{l} \leq \\ < \\ \neq \end{array} \right\} 0 \right)_{\hat{x}}^{-B/(2A)} \equiv \left( A \frac{(-B)^2}{4A^2} + \frac{-B^2}{2A} + C \left\{ \begin{array}{l} \leq \\ < \\ \neq \end{array} \right\} 0 \right) \equiv \left( \frac{-B^2}{4A} + C \left\{ \begin{array}{l} \leq \\ < \\ \neq \end{array} \right\} 0 \right)$$

The formula resulting from Note 11 might be bigger than that of Theorem 6 but it does not increase the polynomial degrees.

Further optimizations are possible if some signs of  $a, b$  are known, because several cases in the quantifier-free expansion then become impossible and can be simplified to *true* or *false* immediately. This helps simplify the formula in Theorem 6, because one of the cases  $a = 0$  versus  $a \neq 0$  might drop. But it also reduces the number of disjuncts in  $F_{\hat{x}}^{-\infty}$ , see Example 2, and in the virtual substitutions of square roots (Lecture 18) and of infinitesimals (Sect. 4), which can lead to significant simplifications.

Theorem 6 also applies for polynomials of higher degrees in  $x$  if all those factor to polynomials of at most quadratic degree in  $x$  [Wei97]. Degree reduction is also possible by renaming based on the greatest common divisor of all powers of  $x$  that occur in  $F$ . If a quantified variable  $x$  occurs only with degrees that are multiples of an odd number  $d$  then virtual substitution can use  $\exists x F(x^d) \equiv \exists y F(y)$ . If  $x$  only occurs with degrees that are multiples of an even number  $d$  then  $\exists x F(x^d) \equiv \exists y (y \geq 0 \wedge F(y))$ . Finally, the cases in Theorem 6 with infinitesimals  $+\varepsilon$  are only needed if  $x$  occurs in strict inequalities in  $F$ . The cases  $F_{\hat{x}}^{(-b \pm \sqrt{d})/(2a)}$  are only needed if  $x$  occurs in equations or weak inequalities.

## 6. Summary

Virtual substitution is one technique for eliminating quantifiers in real arithmetic. It works for linear and quadratic constraints and can be extended to some cubic cases [Wei94]. Virtual substitution can be applied repeatedly from inside out to eliminate quantifiers. In each case, however, virtual substitution requires the eliminated variable to occur with small enough degrees only. Even if that was the case initially, it may stop to be the case after eliminating the innermost quantifier, because the degrees of the formulas resulting from virtual substitution may increase. In that case, degree optimizations and simplifications may sometimes work. If not, then other quantifier elimination techniques need to be used, which are based on semialgebraic geometry or model theory. Virtual substitution alone always works for mixed quadratic-linear formulas, i.e. those in which all quantified variables occur linearly except for one variable that occurs quadratically. In practice, however, many other cases turn out to work well with virtual substitution.

By inspecting Theorem 6 and its optimizations, we also observe that it is interesting to look at only closed sets or only open sets, corresponding to formulas with only  $\leq$  and  $=$  or sets with only  $<$  and  $\neq$  conditions, because half of the cases then drop out of the expansion in Theorem 6. Furthermore, if the formula  $\exists x F$  only mentions strict inequalities  $<$  and disequations  $\neq$ , then all virtual substitutions will involve infinitesimals or infinities. While both are conceptually more demanding than virtual substitutions with mere square root expressions, the advantage is that both infinitesimals and infinities rarely satisfy any equations (except when they are trivial because all coefficients are zero). In that case, most formulas simplify tremendously. That is an indication in the virtual substitution method for a more general phenomenon: existential arithmetic with strict inequalities or, dually, validity of universal arithmetic with weak inequalities, is computationally easier.

### A. Semialgebraic Geometry

The geometric counterpart of polynomial equations or quantifier-free first-order formulas with polynomial equations are affine varieties. The geometric counterpart of first-order formulas of real arithmetic that may mention inequalities are called semialgebraic sets in real algebraic geometry [BCR98, BPR06]. By quantifier elimination, the sets definable with quantifiers is the same as the sets definable without quantifiers. Hence, the formulas of first-order real arithmetic exactly define semialgebraic sets:

**Definition 8** (Semialgebraic Set).  $S \subseteq \mathbb{R}^n$  is an *semialgebraic set* iff it is defined by a finite intersection of polynomial equations and inequalities or any finite union of such sets.

$$S = \bigcup_{i=1}^t \bigcap_{j=1}^s \{x \in \mathbb{R}^n : p(x) \sim 0\} \quad \text{where } \sim \in \{=, \geq, >\}$$

The geometric counterpart of the quantifier elimination result is that semialgebraic sets are closed under projection (the other closure properties are obvious in logic).

**Theorem 9** (Tarski Seidenberg [Tar51, Sei54]). *Semialgebraic sets are closed under finite unions, finite intersections, complements and projection to linear subspaces.*

The semialgebraic sets corresponding to a number of interesting systems of polynomial inequalities are illustrated in Fig. 4.

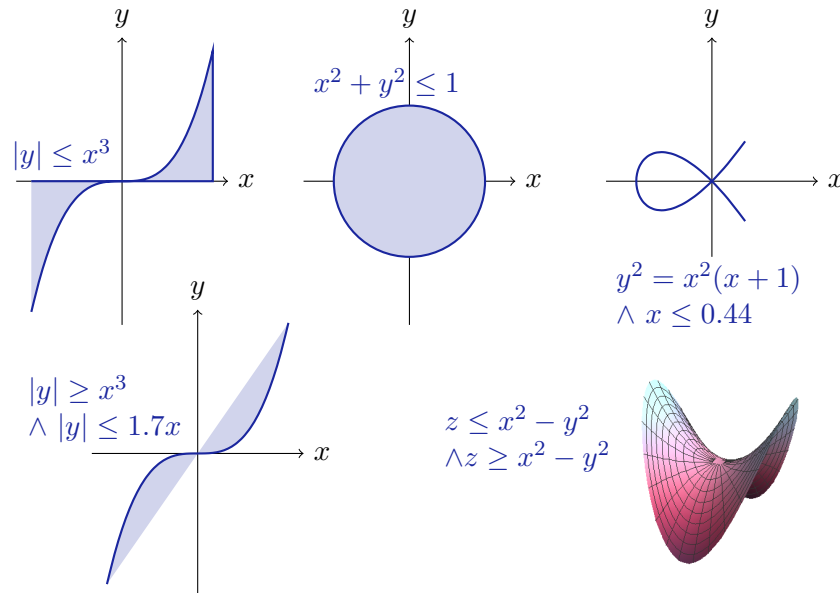


Figure 4: Systems of polynomial inequalities describe semialgebraic sets

### Exercises

Exercise 1. Consider

$$\exists x (ax^2 + bx + c \leq 0 \wedge F) \tag{14}$$

The virtual substitution of the roots of  $ax^2 + bx + c = 0$  according to Sect. 2 as well as of  $-\infty$  according to Sect. 3 will lead to

$$F_{\hat{x}}^{-\infty} \vee a = 0 \wedge b \neq 0 \wedge F_{\hat{x}}^{-c/b} \vee a \neq 0 \wedge b^2 - 4ac \geq 0 \wedge (F_{\hat{x}}^{(-b + \sqrt{b^2 - 4ac})/(2a)} \vee F_{\hat{x}}^{(-b - \sqrt{b^2 - 4ac})/(2a)})$$

But when  $F$  is  $-ax^2 + bx + e < 0$ , then none of those cases necessarily works. Does that mean the result of virtual substitution is not equivalent to (14)? Where is the catch in this argument?

Exercise 2. Perform quantifier elimination by virtual substitution to compute

$$\text{QE}(\exists x (x^2 - x + c \leq 0 \wedge x \geq 0 \wedge -x + 2 \leq 0))$$

*Exercise 3.* Develop and prove a virtual substitution formula for quadratic polynomials analog to Theorem 6 that uses the points illustrated in Fig. 3(right) instead of Fig. 3(left).

## References

- [BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy. *Real Algebraic Geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1998.
- [BD07] Christopher W. Brown and James H. Davenport. The complexity of quantifier elimination and cylindrical algebraic decomposition. In Dongming Wang, editor, *ISSAC*, pages 54–60. ACM, 2007.
- [BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*. Springer, 2nd edition, 2006. doi:10.1007/3-540-33099-2.
- [dMP13] Leonardo Mendonça de Moura and Grant Olney Passmore. Computation in real closed infinitesimal and transcendental extensions of the rationals. In Maria Paola Bonacina, editor, *Automated Deduction - CADE-24 - 24th International Conference on Automated Deduction, Lake Placid, NY, USA, June 9-14, 2013. Proceedings*, volume 7898 of *LNCS*, pages 178–192. Springer, 2013.
- [Pas11] Grant Olney Passmore. *Combined Decision Procedures for Nonlinear Arithmetics, Real and Complex*. PhD thesis, School of Informatics, University of Edinburgh, 2011.
- [Pla08] André Platzer. Differential dynamic logic for hybrid systems. *J. Autom. Reas.*, 41(2):143–189, 2008. doi:10.1007/s10817-008-9103-8.
- [Pla10] André Platzer. *Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics*. Springer, Heidelberg, 2010. doi:10.1007/978-3-642-14509-4.
- [Pla12] André Platzer. Logics of dynamical systems. In *LICS*, pages 13–24. IEEE, 2012. doi:10.1109/LICS.2012.13.
- [PQR09] André Platzer, Jan-David Quesel, and Philipp Rümmer. Real world verification. In Renate A. Schmidt, editor, *CADE*, volume 5663 of *LNCS*, pages 485–501. Springer, 2009. doi:10.1007/978-3-642-02959-2\_35.
- [Sei54] Abraham Seidenberg. A new decision method for elementary algebra. *Annals of Mathematics*, 60:365–374, 1954.
- [Tar51] Alfred Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, Berkeley, 2nd edition, 1951.
- [Wei94] Volker Weispfenning. Quantifier elimination for real algebra — the cubic case. In *ISSAC*, pages 258–263, 1994.
- [Wei97] Volker Weispfenning. Quantifier elimination for real algebra — the quadratic case and beyond. *Appl. Algebra Eng. Commun. Comput.*, 8(2):85–101, 1997.