1 Checking proofs

We now turn to the question of translating proofs to programs and back again. In these notes, we present both for the sake of accessibility.

Task 1. Find a program proving \((A \supset B \supset C) \supset (B \supset A \supset C)\) and proof check its validity.

Solution 1: Program:

\[
fn\ f \Rightarrow fn\ b \Rightarrow fn\ a \Rightarrow (f\ a)\ b
\]

Proof:

\[
\begin{align*}
\dfrac{f : A \supset B \supset C}{fa : B \supset C} & \quad \dfrac{a : A}{a \downarrow \uparrow} \quad \dfrac{\vdash E}{b \downarrow \uparrow} \\
\dfrac{fa : B \supset C}{(f\ a)\ b : C} & \quad \dfrac{(f\ a)\ b : C}{\supset I^a} \\
\dfrac{fn\ a \Rightarrow (f\ a)\ b : A \supset C}{\supset I^a} & \quad \dfrac{fn\ b \Rightarrow fn\ a \Rightarrow (f\ a)\ b : B \supset A \supset C}{\supset I^b} \\
\dfrac{fn\ f \Rightarrow fn\ b \Rightarrow fn\ a \Rightarrow (f\ a)\ b : (A \supset B \supset C) \supset B \supset A \supset C}{\supset I^f}
\end{align*}
\]

Task 2. Find a program proving \(((A \supset B) \lor (A \supset C)) \supset A \supset (B \lor C)\) and proof check its validity.

Solution 2: Program:

\[
fn\ u \Rightarrow fn\ v \Rightarrow \text{case } u \text{ of } inl\ f \Rightarrow \text{inl } (f\ v) \mid \text{inr } g \Rightarrow \text{inr } (g\ v)
\]

Proof:

\[
\begin{align*}
\dfrac{f : A \supset B}{fa : B} & \quad \dfrac{a : A}{a \downarrow \uparrow} \quad \dfrac{\vdash E}{g : A \supset C} \\
\dfrac{fa : B} {ga : C \downarrow \uparrow} & \quad \dfrac{ga : C} {\vdash E/g} \\
\dfrac{u : (A \supset B) \lor (A \supset C)} {\text{inl } (f\ a) : B \lor C} \quad \text{\text{\vdash } I_1} & \quad \dfrac{\text{inr } (g\ a) : B \lor C} {\text{\vdash } I_2} \\
\text{\vdash I^a} & \quad \text{\vdash I^b} \\
\dfrac{fn\ a \Rightarrow \text{case } u \text{ of } inl\ f \Rightarrow \text{inl } (f\ a) \mid \text{inr } g \Rightarrow \text{inr } (g\ a) : A \supset (B \lor C)} {\supset I^a} \\
\dfrac{fn\ u \Rightarrow fn\ a \Rightarrow \text{case } u \text{ of } inl\ f \Rightarrow \text{inl } (f\ a) \mid \text{inr } g \Rightarrow \text{inr } (g\ a) : (A \supset B) \lor (A \supset C) \supset A \supset (B \lor C)} {\supset I^u}
\end{align*}
\]
2 Focusing

Like with chaining (Lecture 14), all propositions have explicit polarity: $A^+, B^-$. Shifts, ↓ $A$, ↑ $B$ are added to the grammar to explicitly change the polarity of a formula. This ensures that all formulas can be used (and not just Horn clauses). During proof search, these shifts induce a switch between the chaining phase and the inversion phase (Lecture 12).

These shifts have the following rules:

- In the inversion phase:

$$
\frac{\Gamma; \Omega^+}{\Gamma; \Omega^+ \rightarrow \uparrow A^+} \quad \uparrow R
$$

$$
\frac{\Gamma, A^{-}; \Omega^+}{\Gamma, A^{-}; \Omega^+ \rightarrow e} \quad \downarrow L
$$

- In the chaining phase:

$$
\frac{\Gamma; \cdot}{\Gamma; \cdot \rightarrow \downarrow A^-} \quad \downarrow R
$$

$$
\frac{\Gamma; A^+}{\Gamma; A^+ \rightarrow e} \quad \uparrow L
$$

Since atomic propositions can also be polarised positively, $P^+$, we need to add a corresponding rule for chaining:

$$
\frac{\Gamma, P^+}{\Gamma, [P^+] \rightarrow \text{id}^+}
$$

All in all, the proof search goes as follows:

- In the inversion phase, right rules can only be applied when the succedent is negative.

- Similarly, left rules can only be applied when an antecedent is positive.

- Right rules are applied before left rules in the inversion phase.

- Once stabilised, the focusR and focusL rules transition a proof from the inversion phase to the chaining phase.

- The ↓R and ↑L rules transition a proof from the chaining phase back to the inversion phase. They result in a loss of focus on the current proposition.

**Task 3.** Pick a polarization of the atoms in $((A \lor B) \supset C) \land A \supset C$ and prove it with the focusing rules.

**Solution 3:** We pick $A^+, B^-$, and $C^-$, which gives us ↓ $(A^+ \lor \downarrow B^-) \supset C^- \land A^+ \supset C^-$. We let $\Gamma_0 = (A^+ \lor \downarrow B^-) \supset C^-, A^+$