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1 All the sequent calculi

We have seen in lecture four different sequent calculi, each improving on the previous for automatic (and, let's be honest, manual) proof search.

1.1 Sequent calculus

First there was sequent calculus, which can be obtained quite straightforwardly from the natural deduction calculus with verification judgments.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C} \supset L$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \Longrightarrow P} \mathit{init} \qquad \frac{}{\Gamma \Longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C} \bot L$$

1.2 Restricted sequent calculus

We quickly realize that the sequent calculus above can't be good for proof search, as it keeps a copy of every formula potentially wasting memory and increasing the search space. So we notice we can restrict it and, in the end, the only formula we actually need to keep copies of are implications on the left.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P} \mathit{init} \qquad \frac{}{\Gamma \longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C} \bot L$$

1.3 Inversion sequent calculus

Playing around with the calculus above, we notice that some rules are *invertible*, meaning that their premises are justified from the conclusion[1]. Therefore we can eagerly apply those rules when doing proof search, without looking back. This reduces the search space considerably, since we don't need to backtrack on every rule application, only on the non-invertible ones.

$$\frac{\Gamma^-; \Omega, A \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \supset B} \supset\!R \qquad \frac{\Gamma^-, A \supset B; \cdot \xrightarrow{R} A \quad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, A \supset B; \cdot \xrightarrow{L} C^+} \supset\!L$$

$$\frac{\Gamma^-; \Omega \xrightarrow{R} A \quad \Gamma^-; \Omega \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \wedge B} \wedge R \qquad \frac{\Gamma^-; \Omega, A, B \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \wedge B \xrightarrow{L} C^+} \wedge L$$

$$\frac{\Gamma^-; \cdot \xrightarrow{R} A}{\Gamma^-; \cdot \xrightarrow{L} A \vee B} \vee R_1 \qquad \frac{\Gamma^-; \cdot \xrightarrow{R} B}{\Gamma^-; \cdot \xrightarrow{L} A \vee B} \vee R_2 \qquad \frac{\Gamma^-; \Omega, A \xrightarrow{L} C^+ \quad \Gamma^-; \Omega, B \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \vee B \xrightarrow{L} C^+} \vee L$$

$$\frac{P \in \Gamma^-}{\Gamma^-; \Omega \xrightarrow{R} P} \,\mathsf{init} \qquad \frac{P = C^+}{\Gamma^-; \Omega, P \xrightarrow{L} C^+} \,\mathsf{init} \qquad \frac{}{\Gamma^-; \Omega \Longrightarrow \top} \top R \qquad \frac{}{\Gamma^-; \Omega, \bot \xrightarrow{L} C^+} \bot L$$

$$\frac{P \notin \Gamma^- \quad \Gamma^-; \Omega \xrightarrow{L} P}{\Gamma^-; \Omega \xrightarrow{R} P} \,\mathsf{LR}_P \qquad \frac{\Gamma^-; \Omega \xrightarrow{L} A \vee B}{\Gamma^-; \Omega \xrightarrow{R} A \vee B} \,\mathsf{LR}_\vee \qquad \frac{\Gamma^-; \Omega \xrightarrow{L} \bot}{\Gamma^-; \Omega \xrightarrow{R} \bot} \,\mathsf{LR}_\bot$$

$$\frac{\Gamma^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, \top \xrightarrow{L} C^+} \top L \qquad \frac{\Gamma^-, P; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, P \xrightarrow{L} C^+} \,\mathsf{shift}_P \qquad \frac{\Gamma^-, A \supset B; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \supset B \xrightarrow{L} C^+} \,\mathsf{shift}_\supset$$

1.4 Contraction-free sequent calculus (a.k.a. G4ip)

Still we have the problem of needing to keep implications on the left around. By analyzing what might happen on the left side of an implication more carefully, we can come up with a calculus where this implicit contraction of implications no longer occurs. This is perfect for proof search and it gives directly a decision procedure for propositional intuitionistic logic (which is good anyway, since this is indeed a decidable fragment).

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset\!R \qquad \frac{P \in \Gamma \quad \Gamma, B \longrightarrow C}{\Gamma, P \supset B \longrightarrow C} P\!\supset\!L \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C} \top\!\supset\!L$$

$$\frac{\Gamma, D \supset E \supset B \longrightarrow C}{\Gamma, D \wedge E \supset B \longrightarrow C} \wedge\!\supset\!L \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C} \bot\!\supset\!L \qquad \frac{\Gamma, D \supset B, E \supset B \longrightarrow C}{\Gamma, D \vee E \supset B \longrightarrow C} \vee\!\supset\!L \qquad \frac{\Gamma, D, E \supset B \longrightarrow E \quad \Gamma, B \longrightarrow C}{\Gamma, (D \supset E) \supset B \longrightarrow C} \supset\!\supset\!L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

[1]The other direction, i.e., the conclusion is justified by the premises, is true for **every** rule.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P}\ \text{init} \qquad \frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

$$\frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

1.5 Exercises

The proposition $\neg\neg(A \vee \neg A)$ was given as an example as to why the rule $\supset L$ must keep the implication in its premise when using the restricted sequent calculus.

Task 1. Prove $\neg\neg(A \vee \neg A)$ in G4ip.

Solution 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \longrightarrow \bot}\ \bot L}{A \supset \bot, A \longrightarrow \bot}\ P \supset L}{\neg A, A, \bot \supset \bot \longrightarrow \bot}\ \bot \supset L \qquad \dfrac{}{\neg A, \bot \longrightarrow \bot}\ \bot L}{\neg A, (A \supset \bot) \supset \bot \longrightarrow \bot}\ \supset\supset L}{\dfrac{(A \vee \neg A) \supset \bot \longrightarrow \bot}{\longrightarrow \neg\neg(A \vee \neg A)}\ \supset L}\ \vee\supset L$$

In the lecture notes it is indicated that cut is admissible for the restricted calculus[2]. The proof is analogous to the one you have already seen, but since less formulas are kept around, some cases become simpler.

Task 2. Prove that if $\Gamma \longrightarrow A \supset B$ and $\Gamma, A \supset B \longrightarrow C$ then $\Gamma \longrightarrow C$ in the restricted sequent calculus (consider only the case where the cut formula is principal).

Solution 2: Assume $\mathcal{D}$ and $\mathcal{E}$ are the following derivations, respectively:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, A \longrightarrow B}}{\Gamma \longrightarrow A \supset B}\ \supset R \qquad \dfrac{\dfrac{\mathcal{E}_1}{\Gamma, A \supset B \longrightarrow A} \quad \dfrac{\mathcal{E}_2}{\Gamma, B \longrightarrow C}}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

$$\begin{array}{rr} \Gamma \longrightarrow A & \text{by IH on } A \supset B, \mathcal{D} \text{ and } \mathcal{E}_1 \\ \Gamma, A \longrightarrow C & \text{by IH on } B, \mathcal{D}_1 \text{ and } \mathcal{E}_2 \\ \Gamma \longrightarrow C & \text{by IH on } A \text{ and both previous lines} \end{array}$$

Task 3. Show that the rules $\wedge \supset L$ and $\vee \supset L$ in G4ip are invertible.

Solution 3:

[2]Actually, cut is admissible for all the calculi listed here.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B, D, E \longrightarrow B}\,\mathit{init}}{E \supset B, D, E \longrightarrow B}\,E \supset L}{D \supset E \supset B, D, E \longrightarrow B}\,D \supset L}{D \supset E \supset B \longrightarrow (D \wedge E) \supset B}\,\supset R, \wedge L \qquad \Gamma, (D \wedge E) \supset B \longrightarrow C}{\Gamma, D \supset E \supset B \longrightarrow C}\,\mathit{cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B, E \supset B, D \longrightarrow B}\,\mathit{init}}{D \supset B, E \supset B, D \longrightarrow B}\,D \supset L \qquad \dfrac{\dfrac{}{D \supset B, B, E \longrightarrow B}\,\mathit{init}}{D \supset B, E \supset B, E \longrightarrow B}\,E \supset L}{D \supset B, E \supset B, D \vee E \longrightarrow B}\,\vee L}{D \supset B, E \supset B \longrightarrow (D \vee E) \supset B}\,\supset R \qquad \Gamma, (D \vee E) \supset B \longrightarrow C}{\Gamma, D \supset B, E \supset B \longrightarrow C}\,\mathit{cut}$$

In G4ip, the invertible rules are:

- $\wedge R$
- $\supset R$
- $\top R$
- $\wedge L$
- $\vee L$
- $\bot L$
- $\top L$
- $\top \supset L$
- $\wedge \supset L$
- $\vee \supset L$
- $\bot \supset L$

When doing a proof in G4ip, or when implementing proof search for G4ip, we should continuously apply these invertible rules until there are none left to apply. Then we must move on to apply the non-invertible rules (also called search rules), which include:

- init
- $\vee R_1$
- $\vee R_2$
- $P \supset L$
- $\supset \supset L$

Task 4. Prove the following sequent in G4ip:

$$\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R$$

Solution 4:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \supset R, (P \supset Q) \supset S, Q \longrightarrow Q}\ \mathit{init}}{P, Q \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow Q}\ P \supset L \qquad \dfrac{}{R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\ \mathit{init}}{(P \supset Q) \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\ \supset\supset L}{(P \supset Q) \supset R, (P \supset Q) \supset S \longrightarrow (P \supset Q) \supset R}\ \supset R}{((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \longrightarrow (P \supset Q) \supset R}\ \wedge L}{\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R}\ \supset R
$$