André Platzer et al

1 Heyting Arithmetic

Now that we have fully explored the surrounding machinery, let's try and look at a more sophisticated system of logic.

$$\dfrac{n\colon \mathsf{nat} \quad C(0)\ \mathit{true} \quad \begin{matrix} \overline{x : \mathsf{nat}} \quad \overline{C(x)\ \mathit{true}}^{\,u} \\ \vdots \\ C(\mathsf{s}\ x)\ \mathit{true} \end{matrix}}{C(n)\ \mathit{true}}\ \mathsf{natE}^{x,u}$$

The other was the *rule of primitive recursion*, which introduces a new term constructor R for each type τ:

$$\dfrac{n\colon \mathsf{nat} \quad t_0\colon \tau \quad \begin{matrix} \overline{x : \mathsf{nat}} \quad \overline{r : \tau} \\ \vdots \\ t_s\colon \tau \end{matrix}}{R(n, t_0, x.\,r.\,t_s)\colon \tau}\ \mathsf{natE}^{x,r}$$

Its behaviour is captured by the following reduction rules:

$$R(0, t_0, x.\,r.\,t_s) \Longrightarrow_R t_0,$$
$$R(\mathsf{s}\ n', t_0, x.\,r.\,t_s) \Longrightarrow_R [R(n', t_0, x.\,r.\,t_s)/r][n'/x]\,t_s.$$

These rules R indicate that R describes a recursive function "$R(n)$" on the first parameter, with value t_0 when $n = 0$, and value $[R(n')/r][n'/x]t_s$ when $n = \mathsf{s}\ n'$. This motivates the more readable *schema of primitive recursion*, where we define the function (call it "f" to avoid confusion) f by cases:

$$f(0) = t_0,$$
$$f(\mathsf{s}\ x) = t_s(x, f(x)).$$

We can recover the recursor version of the definition as follows:

$$f = (\mathsf{fn}\ n \Rightarrow R(n, t_0, x.r.t_s(x, r))).$$

1.1 All of the rules in one place!

Here are all of the Heyting arithmetic rules.

$$\dfrac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \dfrac{x : \mathsf{nat}}{\mathsf{s}\ x : \mathsf{nat}}\ \mathsf{nat}I_s$$

$$\dfrac{x : \mathsf{nat} \quad C(0)\ \mathit{true} \qquad \begin{array}{c}\overline{y : \mathsf{nat}} \quad \overline{C(y)\ \mathit{true}} \\ \vdots \\ C(\mathsf{s}\ y)\ \mathit{true}\end{array}}{C(x)\ \mathit{true}}\ \mathsf{nat}E^{y,u}$$

$$\dfrac{}{0 = 0\ \mathit{true}} = I_{00} \qquad \dfrac{x = y\ \mathit{true}}{\mathsf{s}\ x = \mathsf{s}\ y\ \mathit{true}} = I_{ss}$$

$$\dfrac{0 = \mathsf{s}\ x\ \mathit{true}}{C\ \mathit{true}} = E_{0s} \qquad \dfrac{\mathsf{s}\ x = 0\ \mathit{true}}{C\ \mathit{true}} = E_{s0} \qquad \dfrac{\mathsf{s}\ x = \mathsf{s}\ y\ \mathit{true}}{x = y\ \mathit{true}} = E_{ss}$$

$$\dfrac{}{R(0, t_0, x.r.t_s) \Rightarrow_R t_0} \Rightarrow_R I_0 \qquad \dfrac{}{R(\mathsf{s}\ n, t_0, x.r.t_s) \Rightarrow_R [R(n, t_0, x.r.t_s)/r][n/x]t_s} \Rightarrow_R I_s$$

$$\dfrac{A(x)\ \mathit{true} \quad x \Rightarrow_R y}{A(y)\ \mathit{true}} \Rightarrow_R E_1 \qquad \dfrac{A(y)\ \mathit{true} \quad x \Rightarrow_R y}{A(x)\ \mathit{true}} \Rightarrow_R E_2$$

1.2 Working with these ideas

Task 1. The judgmental form of the principle of induction can be used to show the following more traditional formulation that uses universal quantification:

$$\forall n : \mathsf{nat}.\, C(0) \supset (\forall x : \mathsf{nat}.\, C(x) \supset C(\mathsf{s}\ x)) \supset C(n)\ \mathit{true}.$$

Solution 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n : \mathsf{nat}} \quad \dfrac{}{C(0)}\,u \quad \dfrac{\dfrac{\dfrac{}{\forall x : \mathsf{nat}.\, C(x) \supset C(\mathsf{s}\ x)}\,v \quad \dfrac{}{x : \mathsf{nat}}}{C(x) \supset C(\mathsf{s}\ x)}\,\forall\mathsf{E} \quad \dfrac{}{C(x)}\,w}{C(\mathsf{s}\ x)}\,{\supset}\mathsf{E}}{C(n)}\,\mathsf{natE}^{x,w}}{(\forall x : \mathsf{nat}.\, C(x) \supset C(\mathsf{s}\ x)) \supset C(n)}\,{\supset}\mathsf{I}^{v}}{C(0) \supset (\forall x : \mathsf{nat}.\, C(x) \supset C(\mathsf{s}\ x)) \supset C(n)}\,{\supset}\mathsf{I}^{u}}{\forall n : \mathsf{nat}.\, C(0) \supset (\forall x : \mathsf{nat}.\, C(x) \supset C(\mathsf{s}\ x)) \supset C(n)}\,\forall\mathsf{I}^{n}$$

Task 2. Prove

$$\forall n : \mathsf{nat}.\, R(n, 0, x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(n, n, x.r.\mathsf{s}\ r)\ \mathit{true}$$

You may assume for the purposes of this proof that $R(\mathsf{s}\ y, \mathsf{s}\ y, x.r.\mathsf{s}\ r) \Rightarrow_R \mathsf{s}\ (R(\mathsf{s}\ y, y, x.r.\mathsf{s}\ r))$ (note that while they are equivalent, neither side actually reduces to the other).
Furthermore, *although this is not a rule*, assume you may step underneath successors, as if you have a rule $\Rightarrow_R I^*$ with the premise $x \Rightarrow_R y$ and conclusion $\mathsf{s}\ x \Rightarrow_R \mathsf{s}\ y$.

Solution 2: Let the following section be called X:

$$\dfrac{\dfrac{\dfrac{}{0 = 0\ \mathit{true}} = I_{00} \quad \dfrac{}{R(0, 0, x.r.\mathsf{s}\ (\mathsf{s}\ r)) \Rightarrow_R 0\ \mathit{true}} \Rightarrow_R I_0}{R(0, 0, x.r.\mathsf{s}\ (\mathsf{s}\ r)) = 0\ \mathit{true}} \Rightarrow_R E_2 \quad \dfrac{}{R(0, 0, x.r.\mathsf{s}\ r) \Rightarrow_R 0\ \mathit{true}} \Rightarrow_R I_0}{R(0, 0, x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(0, 0, x.r.\mathsf{s}\ r)\ \mathit{true}} \Rightarrow_R E_2$$

Let the following section be called Y:

$$\dfrac{\dfrac{\dfrac{\overline{R(y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(y,y,x.r.\mathsf{s}\ r)\ true}\ u}{\mathsf{s}\ (R(y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r))) = \mathsf{s}\ (R(y,y,x.r.\mathsf{s}\ r))\ true} = I_{ss}}{\mathsf{s}\ (\mathsf{s}\ (R(y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)))) = \mathsf{s}\ (\mathsf{s}\ (R(y,y,x.r.\mathsf{s}\ r)))\ true} = I_{ss} \quad \dfrac{}{R(\mathsf{s}\ y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) \Rightarrow_R \mathsf{s}\ (\mathsf{s}\ (R(y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r))))\ true} \Rightarrow_R I_s}{R(\mathsf{s}\ y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = \mathsf{s}\ (\mathsf{s}\ (R(y,y,x.r.\mathsf{s}\ r)))} \Rightarrow_R E_2$$

Let the following section be called Z:

$$\dfrac{\dfrac{Y \quad \dfrac{\dfrac{\overline{R(\mathsf{s}\ y,y,x.r.\mathsf{s}\ r) \Rightarrow_R \mathsf{s}\ (R(y,y,x.r.\mathsf{s}\ r))\ true}\ \Rightarrow_R I_s}{\mathsf{s}\ (R(\mathsf{s}\ y,y,x.r.\mathsf{s}\ r)) \Rightarrow_R \mathsf{s}\ (\mathsf{s}\ (R(y,y,x.r.\mathsf{s}\ r)))\ true} \Rightarrow_R I^*}{}}{R(\mathsf{s}\ y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = \mathsf{s}\ (R(\mathsf{s}\ y,y,x.r.\mathsf{s}\ r))\ true} \Rightarrow_R E_2 \quad \dfrac{}{R(\mathsf{s}\ y,\mathsf{s}\ y,x.r.\mathsf{s}\ r) \Rightarrow_R \mathsf{s}\ (R(\mathsf{s}\ y,y,x.r.\mathsf{s}\ r))\ true}\ \text{given}}{R(\mathsf{s}\ y,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(\mathsf{s}\ y,\mathsf{s}\ y,x.r.\mathsf{s}\ r)\ true} \Rightarrow_R E_2$$

The full proof is then:

$$\dfrac{\dfrac{\overline{a\colon nat} \quad X \quad Z}{R(a,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(a,a,x.r.\mathsf{s}\ r)\ true}\ \mathsf{natE}^{y,u}}{\forall n : \mathsf{nat}.\, R(n,0,x.r.\mathsf{s}\ (\mathsf{s}\ r)) = R(n,n,x.r.\mathsf{s}\ r)\ true}\ \forall\mathsf{I}^n$$

2 Complete induction

2.1 Inequality

Similarly to equality, there are multiple ways to formally define inequality. Here, we only focus on $\leq$. The introduction rules are defined on the introduction rules for natural numbers. 0 is less or equal to any number ($\leq I_{0n}$) and taking the succesors does not break the order ($\leq I_{ss}$). Compared to the case of equality, we have one less elimination rule. There is no equivalent to $= I_{0s}$ as $0 \leq \mathsf{s}\ x$ does not contain any information.

$$\dfrac{}{0 \leq x\ true} \leq I_{0x} \qquad \dfrac{x \leq y\ true}{\mathsf{s}\ x \leq \mathsf{s}\ y\ true} \leq I_{ss}$$

$$\dfrac{\mathsf{s}\ x \leq 0\ true}{C\ true} \leq E_{s0} \qquad \dfrac{\mathsf{s}\ x \leq \mathsf{s}\ y\ true}{x \leq y\ true} \leq E_{ss}$$

Task 3. Prove that inequality is reflexive, i.e.

$$\forall n : nat.\, n \leq n\ true$$

Solution 3:

$$\dfrac{\dfrac{\overline{n\colon \mathsf{nat}} \quad \dfrac{}{0 \leq 0\ true} \leq I_{0x} \quad \dfrac{\overline{x \leq x\ true}\ w}{\mathsf{s}\ x \leq \mathsf{s}\ x\ true} \leq I_{ss}}{n \leq n\ true}\ \mathsf{natE}^{x,w}}{\forall n : nat.\, n \leq n\ true}\ \forall\mathsf{I}^n$$

Task 4. Prove that inequality is transitive, i.e.

$$\forall m : nat.\, \forall n : nat.\, \forall o : nat.\, m \leq n \supset n \leq o \supset m \leq o\ true$$

Solution 4: We write $C(x)$ for the formula $\forall n : nat.\, \forall o : nat.\, x \leq n \supset n \leq o \supset x \leq o\ \ true$. We thus want to prove $\forall m : \mathsf{nat}.\, C(m)$.

Let the following section be called Z:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C(x)\ \mathit{true}}u \quad \dfrac{}{y\colon \mathsf{nat}}}{\forall o : \mathsf{nat}.\, x \le y \supset y \le o \supset x \le o\ \mathit{true}}\forall\mathsf{E} \quad \dfrac{}{z\colon \mathsf{nat}}}{x \le y \supset y \le z \supset x \le z\ \mathit{true}}\forall\mathsf{E} \quad \dfrac{\dfrac{}{\mathsf{s}\,x \le \mathsf{s}\,y\ \mathit{true}}v'}{x \le y\ \mathit{true}}\le E_{ss}}{y \le z \supset x \le z\ \mathit{true}}\supset\mathsf{E} \quad \dfrac{\dfrac{}{\mathsf{s}\,y \le \mathsf{s}\,z\ \mathit{true}}w'}{y \le z\ \mathit{true}}\le E_{ss}}{x \le z\ \mathit{true}}\supset\mathsf{E}$$

Let the following section be called Y:

$$\dfrac{\dfrac{}{o\colon \mathsf{nat}} \quad \dfrac{\dfrac{\dfrac{}{\mathsf{s}\,y \le 0\ \mathit{true}}w}{\mathsf{s}\,x \le 0\ \mathit{true}}\le E_{s0}}{\mathsf{s}\,y \le 0 \supset \mathsf{s}\,x \le 0\ \mathit{true}}\supset\mathsf{I}^{w} \quad \dfrac{\dfrac{\begin{matrix}\mathrm{Z}\\ x \le z\ \mathit{true}\end{matrix}}{\mathsf{s}\,x \le \mathsf{s}\,z\ \mathit{true}}\le I_{ss}}{\mathsf{s}\,y \le \mathsf{s}\,z \supset \mathsf{s}\,x \le \mathsf{s}\,z\ \mathit{true}}\supset\mathsf{I}^{w'}}{\mathsf{s}\,y \le o \supset \mathsf{s}\,x \le o\ \mathit{true}}\mathsf{natE}^{z,-}$$

Let the following section be called X:

$$\dfrac{\dfrac{\dfrac{}{n\colon \mathsf{nat}} \quad \dfrac{\dfrac{\dfrac{}{\mathsf{s}\,x \le 0\ \mathit{true}}v}{0 \le o \supset \mathsf{s}\,x \le o\ \mathit{true}}\le E_{s0}}{\mathsf{s}\,x \le 0 \supset 0 \le o \supset \mathsf{s}\,x \le o\ \mathit{true}}\supset\mathsf{I}^{v} \quad \dfrac{\begin{matrix}\mathrm{Y}\\ \mathsf{s}\,y \le o \supset \mathsf{s}\,x \le o\ \mathit{true}\end{matrix}}{\mathsf{s}\,x \le \mathsf{s}\,y \supset \mathsf{s}\,y \le o \supset \mathsf{s}\,x \le o\ \mathit{true}}\supset\mathsf{I}^{v'}}{\mathsf{s}\,x \le n \supset n \le o \supset \mathsf{s}\,x \le o\ \mathit{true}}\mathsf{natE}^{y,-}}{C(\mathsf{s}\,x)\ \mathit{true}}\forall\mathsf{I}^{n}, \forall\mathsf{I}^{o}$$

The full proof is then:

$$\dfrac{\dfrac{\dfrac{}{m\colon \mathsf{nat}} \quad \dfrac{\dfrac{}{0 \le o\ \mathit{true}}\le I_{0x}}{C(0)\ \mathit{true}}\forall\mathsf{I}^{-}, \forall\mathsf{I}^{o}, \supset\mathsf{I}^{-}, \supset\mathsf{I}^{-} \quad \begin{matrix}\mathrm{X}\\ C(\mathsf{s}\,x)\ \mathit{true}\end{matrix}}{C(m)\ \mathit{true}}\mathsf{natE}^{x,u}}{\forall m : \mathit{nat}.\, C(m)\ \mathit{true}}\forall\mathsf{I}^{m}$$

2.2 Complete induction

Task 5. Prove by mathematical induction that complete induction is sound, i.e.

$$\forall x : \mathsf{nat}.\, (\forall m : \mathsf{nat}.\, (\mathsf{s}\ m \le x \supset C(m)) \supset C(x)) \supset \forall n : \mathsf{nat}.\, C(n)$$

You may use directly that inequality is reflexive and transitive.

Solution 5: We assume $\forall x : \mathsf{nat}.\, (\forall m : \mathsf{nat}.\, (\mathsf{s}\ m \le x \supset C(m)) \supset C(x))$. We will prove a stronger induction property: $\forall k : \mathsf{nat}.\, (k \le n \supset C(k))$. It is easy to see that it implies $C(n)$ by reflexivity of inequality.

- **Base:** $n = 0$. We distinguish whether k is 0 or not.
  - if $k = 0$, by $\forall\mathsf{I}$, $\supset\mathsf{I}$ and $\le E_{s0}$, we prove $\forall m : \mathsf{nat}.\, (\mathsf{s}\ m \le 0 \supset C(m))$.

$\forall x : \mathsf{nat}.\, (\forall m : \mathsf{nat}.\, (\mathsf{s}\ m \le x \supset C(m)) \supset C(x))$	by assumption
$\forall m : \mathsf{nat}.\, (\mathsf{s}\ m \le 0 \supset C(m)) \supset C(0)$	by $\forall\mathsf{E}$
$C(0)$	by $\supset\mathsf{E}$
$0 \le 0 \supset C(0)$	by $\supset\mathsf{I}$

  - if $k = \mathsf{s}\ k'$, then we prove $\mathsf{s}\ k' \le 0 \supset C(k)$ by $\supset\mathsf{I}$ and $\le E_{s0}$.
- **IH:** $\forall k : \mathsf{nat}.\, (k \le n' \supset C(k))$
- **Step:** $n = \mathsf{s}\ n'$. Again, we distinguish whether k is 0 or not.
  - if $k = 0$, we redo the same proof as before to show $C(0)$. Then, by using $\supset\mathsf{I}$, we have $0 \le n \supset C(0)$.

– if $k = \mathsf{s}\,k'$, $\mathsf{s}\,k' \leq \mathsf{s}\,n'$ implies $k' \leq n'$ by $\leq E_{ss}$. We prove first $\forall m : \mathsf{nat}.\,(\mathsf{s}\,m \leq \mathsf{s}\,k' \supset C(m))$.

$$\begin{array}{lr}
\mathsf{s}\,m \leq \mathsf{s}\,k' & \text{by assumption } u \\
m \leq k' & \text{by } \leq E_{ss} \\
m \leq n' & \text{by transitivity} \\
C(m) & \text{by IH} \\
\mathsf{s}\,m \leq \mathsf{s}\,k' \supset C(m) & \text{by } {\supset}\mathsf{I}^u \\
\forall m : \mathsf{nat}.\,(\mathsf{s}\,m \leq \mathsf{s}\,k' \supset C(m)) & \text{by } \forall\mathsf{I}^m
\end{array}$$

We can then apply our original assumption and conclude:

$$\begin{array}{lr}
\forall x : \mathsf{nat}.\,(\forall m : \mathsf{nat}.\,(\mathsf{s}\,m \leq x \supset C(m)) \supset C(x)) & \text{by assumption} \\
\forall m : \mathsf{nat}.\,(\mathsf{s}\,m \leq \mathsf{s}\,k' \supset C(m)) \supset C(\mathsf{s}\,k') & \text{by } \forall\mathsf{E} \\
C(\mathsf{s}\,k') & \text{by } {\supset}\mathsf{E} \\
\mathsf{s}\,k' \leq \mathsf{s}\,n' \supset C(\mathsf{s}\,k') & \text{by } {\supset}\mathsf{I}
\end{array}$$

3 Warshall algorithm

Apart from natural numbers, other inductive objects can be defined, like *lists*. Here we describe the rules for a list of elements of type τ.

$$\frac{}{nil : \tau\ \mathsf{list}}\ \mathsf{listI}_{nil} \qquad \frac{x : \tau \quad p : \tau\ \mathsf{list}}{x :: p : \tau\ \mathsf{list}}\ \mathsf{listI}_{cons}$$

$$\frac{p : \mathsf{list} \quad C(nil)\ \mathit{true} \quad \begin{array}{c} \overline{x : \tau} \quad \overline{q : \tau\ \mathsf{list}} \quad \overline{C(q)\ \mathit{true}}^{\,u} \\ \vdots \\ C(x :: q)\ \mathit{true} \end{array}}{C(p)\ \mathit{true}}\ \mathsf{listE}^{x,q,u}$$

Having this new data structure at hand, we can reason about more complex properties. We refer to the example on Warshall's algorithm in the lecture notes for more details on that matter.

Constructive Logic (15-317), Spring 2021
Recitation 8: Playing with proof systems (2021-03-24)
Evan Cavallo, Oliver Daids, Giselle Reis

1 All the sequent calculi

We have seen in lecture four different sequent calculi, each improving on the previous for automatic (and, let's be honest, manual) proof search.

1.1 Sequent calculus

First there was sequent calculus, which can be obtained quite straightforwardly from the natural deduction calculus with verification judgments.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C} \supset L$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \Longrightarrow P}\, init \qquad \frac{}{\Gamma \Longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C} \bot L$$

1.2 Restricted sequent calculus

We quickly realize that the sequent calculus above can't be good for proof search, as it keeps a copy of every formula potentially wasting memory and increasing the search space. So we notice we can restrict it and, in the end, the only formula we actually need to keep copies of are implications on the left.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P}\, init \qquad \frac{}{\Gamma \longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C} \bot L$$

1.3 Inversion sequent calculus

Playing around with the calculus above, we notice that some rules are *invertible*, meaning that their premises are justified from the conclusion[1]. Therefore we can eagerly apply those rules when doing proof search, without looking back. This reduces the search space considerably, since we don't need to backtrack on every rule application, only on the non-invertible ones.

$$\frac{\Gamma^-;\Omega, A \xrightarrow{R} B}{\Gamma^-;\Omega \xrightarrow{R} A \supset B}\ {\supset}R \qquad \frac{\Gamma^-, A\supset B; \cdot \xrightarrow{R} A \quad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, A \supset B; \cdot \xrightarrow{L} C^+}\ {\supset}L$$

$$\frac{\Gamma^-;\Omega \xrightarrow{R} A \quad \Gamma^-;\Omega \xrightarrow{R} B}{\Gamma^-;\Omega \xrightarrow{R} A \wedge B}\ {\wedge}R \qquad \frac{\Gamma^-;\Omega, A, B \xrightarrow{L} C^+}{\Gamma^-;\Omega, A\wedge B \xrightarrow{L} C^+}\ {\wedge}L$$

$$\frac{\Gamma^-;\cdot \xrightarrow{R} A}{\Gamma^-;\cdot \xrightarrow{L} A \vee B}\ {\vee}R_1 \qquad \frac{\Gamma^-;\cdot \xrightarrow{R} B}{\Gamma^-;\cdot \xrightarrow{L} A \vee B}\ {\vee}R_2 \qquad \frac{\Gamma^-;\Omega, A \xrightarrow{L} C^+ \quad \Gamma^-;\Omega, B \xrightarrow{L} C^+}{\Gamma^-;\Omega, A \vee B \xrightarrow{L} C^+}\ {\vee}L$$

$$\frac{P \in \Gamma^-}{\Gamma^-;\Omega \xrightarrow{R} P}\ \mathsf{init} \qquad \frac{P = C^+}{\Gamma^-;\Omega, P \xrightarrow{L} C^+}\ \mathsf{init} \qquad \frac{}{\Gamma^-;\Omega \Longrightarrow \top}\ {\top}R \qquad \frac{}{\Gamma^-;\Omega, \bot \xrightarrow{L} C^+}\ {\bot}L$$

$$\frac{P \notin \Gamma^- \quad \Gamma^-;\Omega \xrightarrow{L} P}{\Gamma^-;\Omega \xrightarrow{R} P}\ \mathsf{LR}_P \qquad \frac{\Gamma^-;\Omega \xrightarrow{L} A \vee B}{\Gamma^-;\Omega \xrightarrow{R} A \vee B}\ \mathsf{LR}_\vee \qquad \frac{\Gamma^-;\Omega \xrightarrow{L} \bot}{\Gamma^-;\Omega \xrightarrow{R} \bot}\ \mathsf{LR}_\bot$$

$$\frac{\Gamma^-;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega, \top \xrightarrow{L} C^+}\ {\top}L \qquad \frac{\Gamma^-, P;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega, P \xrightarrow{L} C^+}\ \mathsf{shift}_P \qquad \frac{\Gamma^-, A\supset B;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega, A\supset B \xrightarrow{L} C^+}\ \mathsf{shift}_\supset$$

1.4 Contraction-free sequent calculus (a.k.a. G4ip)

Still we have the problem of needing to keep implications on the left around. By analyzing what might happen on the left side of an implication more carefully, we can come up with a calculus where this implicit contraction of implications no longer occurs. This is perfect for proof search and it gives directly a decision procedure for propositional intuitionistic logic (which is good anyway, since this is indeed a decidable fragment).

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ {\supset}R \qquad \frac{P \in \Gamma \quad \Gamma, B \longrightarrow C}{\Gamma, P \supset B \longrightarrow C}\ P{\supset}L \qquad \frac{\Gamma, B \longrightarrow C}{\Gamma, \top \supset B \longrightarrow C}\ {\top}{\supset}L$$

$$\frac{\Gamma, D \supset E \supset B \longrightarrow C}{\Gamma, D \wedge E \supset B \longrightarrow C}\ {\wedge}{\supset}L \qquad \frac{\Gamma \longrightarrow C}{\Gamma, \bot \supset B \longrightarrow C}\ {\bot}{\supset}L \qquad \frac{\Gamma, D \supset B, E \supset B \longrightarrow C}{\Gamma, D \vee E \supset B \longrightarrow C}\ {\vee}{\supset}L \qquad \frac{\Gamma, D, E \supset B \longrightarrow E \quad \Gamma, B \longrightarrow C}{\Gamma, (D \supset E) \supset B \longrightarrow C}\ {\supset}{\supset}L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ {\wedge}R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ {\wedge}L$$

[1]The other direction, i.e., the conclusion is justified by the premises, is true for **every** rule.

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P}\ \text{init} \qquad \frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

$$\frac{\Gamma \longrightarrow C}{\Gamma, \top \longrightarrow C}\ \top L$$

1.5 Exercises

The proposition $\neg\neg(A \vee \neg A)$ was given as an example as to why the rule $\supset L$ must keep the implication in its premise when using the restricted sequent calculus.

Task 1. Prove $\neg\neg(A \vee \neg A)$ in G4ip.

Solution 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \longrightarrow \bot}\ \bot L}{A \supset \bot, A \longrightarrow \bot}\ P \supset L}{\neg A, A, \bot \supset \bot \longrightarrow \bot}\ \bot \supset L \qquad \dfrac{}{\neg A, \bot \longrightarrow \bot}\ \bot L}{\neg A, (A \supset \bot) \supset \bot \longrightarrow \bot}\ \supset\supset L}{(A \vee \neg A) \supset \bot \longrightarrow \bot}\ \vee\supset L}{\longrightarrow \neg\neg(A \vee \neg A)}\ \supset L$$

In the lecture notes it is indicated that cut is admissible for the restricted calculus[2]. The proof is analogous to the one you have already seen, but since less formulas are kept around, some cases become simpler.

Task 2. Prove that if $\Gamma \longrightarrow A \supset B$ and $\Gamma, A \supset B \longrightarrow C$ then $\Gamma \longrightarrow C$ in the restricted sequent calculus (consider only the case where the cut formula is principal).

Solution 2: Assume $\mathcal{D}$ and $\mathcal{E}$ are the following derivations, respectively:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, A \longrightarrow B}}{\Gamma \longrightarrow A \supset B}\ \supset R \qquad \dfrac{\dfrac{\mathcal{E}_1}{\Gamma, A \supset B \longrightarrow A} \quad \dfrac{\mathcal{E}_2}{\Gamma \longrightarrow B}}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

$\Gamma \longrightarrow A$	by IH on $A \supset B$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma, A \longrightarrow C$	by IH on B, $\mathcal{D}_1$ and $\mathcal{E}_2$
$\Gamma \longrightarrow C$	by IH on A and both previous lines

Task 3. Prove the following sequent in G4ip:

$$\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R$$

[2]Actually, cut is admissible for all the calculi listed here.

Solution 3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \supset R, (P \supset Q) \supset S, Q \longrightarrow Q}\ \mathit{init}}{P, Q \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow Q}\ P \supset L \qquad \dfrac{}{R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\ \mathit{init}}{(P \supset Q) \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\ {\supset}{\supset} L}{(P \supset Q) \supset R, (P \supset Q) \supset S \longrightarrow (P \supset Q) \supset R}\ \supset R}{((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \longrightarrow (P \supset Q) \supset R}\ \wedge L}{\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R}\ \supset R
$$