Constructive Logic (15-317), Spring 2021
Recitation 4: Quantifiers (2021-02-24)
André Platzer et al

1 Quantifiers

Up to now, we have been vague about what, exactly, our atomic propositions A are representing. In order to discuss quantification, however, we need to be precise over what, exactly, we are quantifying over. We do this via a new judgment $t : \tau$, where τ is some to-be-defined type. Oftentimes, we are interested in some particular type, like the type of natural numbers or the type of Turing Machines, but the meaning of the $\exists$ and $\forall$ connectives are independent of this.
The rules for verifying these are as follows:

$$\dfrac{\begin{array}{c}\overline{a : A}\\ \vdots \\ A(a)\uparrow\end{array}}{\forall x : \tau.\, A(x)\uparrow}\,\forall\mathsf{I}^a \qquad \dfrac{\forall x : \tau.\, A(x)\downarrow \quad t : \tau}{A(t)\downarrow}\,\forall\mathsf{E}$$

$$\dfrac{t : \tau \quad A(t)\uparrow}{\exists x : \tau.\, A(x)\uparrow}\,\exists\mathsf{I} \qquad \dfrac{\exists x : \tau.\, A(x)\downarrow \quad \begin{array}{c}\overline{a : \tau}\quad \overline{A(a)\downarrow}^{\,u}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\,\exists\mathsf{E}^{a,u}$$

By now, you should be comfortable with erasing the arrows to recover the rules defining these connectives for natural deduction. The intuition for these rules should be straightforward – to prove that some proposition $A(x)$ is true for all $x : \tau$, we should be able to derive $A(c)$ *true* for some arbitrary $c : \tau$. Similarly, we can introduce an existential by demonstrating some object satisfying the proposition.

Eliminating foralls is similarly simple. To eliminate an existential, however, we must do a little more work. If we have $\exists x : \tau.\, A(x)$, then we may not assume anything else about the witness! It must be an object of type τ, and also that it satisfies $A(x)$, but any other properties must be abstracted out, to be replaced with an arbitrary object with the known properties.

2 Examples with quantifiers

Consider predicates $A(x)$ and $B(x)$ which depend on $x : \tau$.

Task 1. Show $\forall x : \tau.\, A(x) \wedge B(x) \supset \forall x : \tau.\, A(x) \wedge \forall x : \tau.\, B(x)$ *true.*

Solution 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x : \tau.\, A(x)\wedge B(x)\ \mathit{true}}^{\,p} \quad \overline{u : \tau}}{A(u)\wedge B(u)\ \mathit{true}}\,\forall\mathsf{E}}{A(u)\ \mathit{true}}\,\wedge\mathsf{E}_1}{\forall x : \tau.\, A(x)\ \mathit{true}}\,\forall\mathsf{I}_u \quad \dfrac{\dfrac{\dfrac{\overline{\forall x : \tau.\, A(x)\wedge B(x)\ \mathit{true}}^{\,p} \quad \overline{v : \tau}}{A(v)\wedge B(v)\ \mathit{true}}\,\forall\mathsf{E}}{A(v)\ \mathit{true}}\,\wedge\mathsf{E}_2}{\forall x : \tau.\, B(x)\ \mathit{true}}\,\forall\mathsf{I}_v}{\forall x : \tau.\, A(x)\wedge\forall x : \tau.\, B(x)\ \mathit{true}}\,\wedge\mathsf{I}}{\forall x : \tau.\, A(x)\wedge B(x)\supset\forall x : \tau.\, A(x)\wedge\forall x : \tau.\, B(x)\ \mathit{true}}\,\supset\mathsf{I}^p$$

Next, let $A(x, y)$ be a formula with two variables $x : \tau$ and $y : \sigma$.

Task 2. Show that you can “swap” an existential and universal. Do a verification proof.

Solution 2:

$$
\dfrac{\dfrac{\dfrac{}{\exists y : \tau.\, \forall x : \sigma.\, A(x,y) \downarrow}\, u \qquad \dfrac{\dfrac{}{d : \tau} \qquad \dfrac{\dfrac{\dfrac{}{\forall x : \sigma.\, A(x,d) \downarrow}\, v \qquad \dfrac{}{c : \sigma}}{A(c,d) \downarrow}\, \forall\mathsf{E}}{A(c,d) \uparrow}\, \updownarrow}{\exists y : \tau.\, A(c,y) \uparrow}\, \exists\mathsf{I}}{\exists y : \tau.\, A(c,y) \uparrow}\, \exists\mathsf{E}^{d,v}}{\dfrac{}{\forall x : \sigma.\, \exists y : \tau.\, A(x,y) \uparrow}}
$$

$$
\dfrac{\dfrac{\exists y : \tau.\, A(c,y) \uparrow}{\forall x : \sigma.\, \exists y : \tau.\, A(x,y) \uparrow}\, \forall\mathsf{I}^{c}}{(\exists y : \tau.\, \forall x : \sigma.\, A(x,y)) \supset (\forall x : \sigma.\, \exists y : \tau.\, A(x,y)) \uparrow}\, \supset\mathsf{I}^{u}
$$

Constructive Logic (15-317), Spring 2021
Recitation 7: Sequent Calculus and Cut Elimination (2021-03-17)
André Platzer et al

1 The Rules

Recall that left rules correspond to "upside down elimination rules" and that right rules correspond to introduction rules.

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\,\wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\,\wedge L_2 \qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\,\wedge R$$

$$\frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\,\vee L \qquad \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\,\vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\,\vee R_2$$

No $\top L$. $\quad \frac{}{\Gamma \Longrightarrow \top}\,\top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\,\bot L \quad$ No $\bot R$.

$$\frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\,\supset L \qquad \frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\,\supset R$$

$$\frac{}{\Gamma, A \Longrightarrow A}\,\mathsf{id}$$

2 Some Example Proofs

Task 1. $\cdot \Longrightarrow A \supset A$

Solution 1:

$$\frac{\frac{}{A \Longrightarrow A}\,\mathsf{id}}{\cdot \Longrightarrow A \supset A}\,\supset R$$

Task 2. $\cdot \Longrightarrow A \wedge B \supset B \wedge A$

Solution 2:

$$\frac{\frac{\frac{\frac{}{A \wedge B, B \Longrightarrow B}\,\mathsf{id}}{A \wedge B \Longrightarrow B}\,\wedge L_2 \quad \frac{\frac{}{A \wedge B, A \Longrightarrow A}\,\mathsf{id}}{A \wedge B \Longrightarrow A}\,\wedge L_1}{A \wedge B \Longrightarrow B \wedge A}\,\wedge R}{\cdot \Longrightarrow A \wedge B \supset B \wedge A}\,\supset R$$

Task 3. $\cdot \Longrightarrow (A \supset (B \wedge C)) \supset (A \supset B)$

Solution 3:

$$\frac{\frac{\frac{\frac{}{(A \supset (B \wedge C)), A \Longrightarrow A}\,\mathsf{id} \quad \frac{\frac{}{(A \supset (B \wedge C)), A, B \wedge C, B \Longrightarrow B}\,\mathsf{id}}{(A \supset (B \wedge C)), A, B \wedge C \Longrightarrow B}\,\wedge L_1}{(A \supset (B \wedge C)), A \Longrightarrow B}\,\supset L}{(A \supset (B \wedge C)) \Longrightarrow (A \supset B)}\,\supset R}{\cdot \Longrightarrow (A \supset (B \wedge C)) \supset (A \supset B)}\,\supset R$$

Task 4. $\cdot \Longrightarrow (A \supset B \supset C) \supset B \supset A \supset C$

Solution 4:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \supset B \supset C, B, A \Longrightarrow A}\,\text{id} \qquad \cfrac{\cfrac{}{A \supset B \supset C, B, A, B \supset C \Longrightarrow B}\,\text{id} \qquad \cfrac{}{A \supset B \supset C, B, A, B \supset C, C \Longrightarrow C}\,\text{id}}{A \supset B \supset C, B, A, B \supset C \Longrightarrow C}\,{\supset}L}{A \supset B \supset C, B, A \Longrightarrow C}\,{\supset}L}{A \supset B \supset C, B \Longrightarrow A \supset C}\,{\supset}R}{A \supset B \supset C \Longrightarrow B \supset A \supset C}\,{\supset}R}{\cdot \Longrightarrow (A \supset B \supset C) \supset B \supset A \supset C}\,{\supset}R
$$

Task 5. $\cdot \Longrightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))$

Solution 5:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{(A \supset B), (A \wedge C), A \Longrightarrow A}\,\text{id}}{(A \supset B), (A \wedge C) \Longrightarrow A}\,\wedge L_1 \qquad \cfrac{}{(A \supset B), (A \wedge C), B \Longrightarrow B}\,\text{id}}{(A \supset B), (A \wedge C) \Longrightarrow B}\,\supset L \qquad \cfrac{\cfrac{}{(A \supset B), (A \wedge C), C \Longrightarrow C}\,\text{id}}{(A \supset B), (A \wedge C) \Longrightarrow C}\,\wedge L_2}{(A \supset B), (A \wedge C) \Longrightarrow B \wedge C}\,\wedge R}{(A \supset B) \Longrightarrow ((A \wedge C) \supset (B \wedge C))}\,\supset R}{\cdot \Longrightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))}\,\supset R
$$

3 Cuts

As a reminder, the cut theorem is as follows: If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$, then $\Gamma \Longrightarrow C$, where A and C are arbitrary propositions.

In class, we saw portions of the proof of admissibility for the cut rule.

Task 6. Finish the case for the proof of admissibility of cut where $\mathcal{E}$ ends in $\supset R$, and A is not the principal formula of the last inference in $\mathcal{E}$.

Solution 6: We have that

$$\mathcal{D} = \Gamma \Longrightarrow A$$

and

$$\mathcal{E} = \qquad \cfrac{\begin{matrix} E_1 \\ \Gamma, A, C_1 \Longrightarrow C_2 \end{matrix}}{\Gamma, A \Longrightarrow C_1 \supset C_2}\,\supset R$$

$$
\begin{array}{rl}
C = C_1 \supset C_2 & \text{this case} \\
\Gamma, C_1 \Longrightarrow A & \text{weakening of } \mathcal{D} \\
\Gamma, C_1 \Longrightarrow C_2 & \text{IH on } A \text{, weakening of } \mathcal{D} \text{, and } \mathcal{E}_1 \\
\Gamma \Longrightarrow C_1 \supset C_2 & \text{by rule } \supset R \text{ on above}
\end{array}
$$

Task 7. What would the derivations $\mathcal{D}$ and $\mathcal{E}$ look like if we wanted to do the same case as above, but with $\supset L$ instead of $\supset R$ as the last derivation in $\mathcal{E}$?

Solution 7:

$$\mathcal{D} = \Gamma \Longrightarrow A$$

$$\mathcal{E} = \qquad \cfrac{\begin{matrix} E_1 \\ \Gamma', B_1 \supset B_2, A \Longrightarrow B_1 \end{matrix} \quad \begin{matrix} E_2 \\ \Gamma', B_1 \supset B_2, A, B_2 \Longrightarrow C \end{matrix}}{\Gamma', B_1 \supset B_2, A \Longrightarrow C}\,\supset L$$