1 Quantifiers

Up to now, we have been vague about what, exactly, our atomic propositions \( A \) are representing. In order to discuss quantification, however, we need to be precise over what, exactly, we are quantifying over. We do this via a new judgment \( t : \tau \), where \( \tau \) is some to-be-defined type. Oftentimes, we are interested in some particular type, like the type of natural numbers or the type of Turing Machines, but the meaning of the \( \exists \) and \( \forall \) connectives are independent of this.

The rules for verifying these are as follows:

\[
\begin{align*}
\frac{a : A}{\forall x : \tau. A(x)} & \quad \forall I^a \\
\frac{\forall x : \tau. A(x) \downarrow}{A(t) \downarrow} & \quad \forall E \\
\frac{t : \tau \quad A(t) \uparrow}{\exists x : \tau. A(x) \uparrow} & \quad \exists I \\
\frac{\exists x : \tau. A(x) \downarrow}{C \uparrow} & \quad \exists E^a,u
\end{align*}
\]

By now, you should be comfortable with erasing the arrows to recover the rules defining these connectives for natural deduction. The intuition for these rules should be straightforward – to prove that some proposition \( A(x) \) is true for all \( x : \tau \), we should be able to derive \( A(c) \) true for some arbitrary \( c : \tau \). Similarly, we can introduce an existential by demonstrating some object satisfying the proposition.

Eliminating foralls is similarly simple. To eliminate an existential, however, we must do a little more work. If we have \( \exists x : \tau. A(x) \), then we may not assume anything else about the witness! It must be an object of type \( \tau \), and also that it satisfies \( A(x) \), but any other properties must be abstracted out, to be replaced with an arbitrary object with the known properties.

2 Examples with quantifiers

Consider predicates \( A(x) \) and \( B(x) \) which depend on \( x : \tau \).

Task 1. Show \( \forall x : \tau. A(x) \land B(x) \supset \forall x : \tau. A(x) \land \forall x : \tau. B(x) \) true.

Solution 1:

\[
\begin{align*}
\frac{\forall x : \tau. A(x) \land B(x) \text{ true}}{p} & \quad \lor E \\
\frac{u : \tau}{A(u) \text{ true} \quad \land E_1} & \quad \forall E \\
\frac{A(u) \text{ true}}{\forall x : \tau. A(x) \text{ true} \quad \forall I_u} & \quad \forall E \\
\frac{\forall x : \tau. A(x) \land \forall x : \tau. B(x) \text{ true} \quad \land E_2 \quad \forall E}{\forall x : \tau. A(x) \land \forall x : \tau. B(x) \text{ true} \quad \lor I^p}
\end{align*}
\]

Next, let \( A(x, y) \) be a formula with two variables \( x : \tau \) and \( y : \sigma \).

Task 2. Show that you can “swap” an existential and universal. Do a verification proof.
Solution 2:

\[
\forall x : \sigma. A(x, d) \Downarrow \exists \ y : \tau. A(c, y) \Uparrow \ \forall E
\]

\[
\exists y : \tau. \forall x : \sigma. A(x, y) \Downarrow u \exists y : \tau. A(c, y) \Uparrow \ \exists I
\]

\[
\forall x : \sigma. \exists y : \tau. A(x, y) \Uparrow \ \forall I^c
\]

\[
(\exists y : \tau. \forall x : \sigma. A(x, y)) \supset (\forall x : \sigma. \exists y : \tau. A(x, y)) \Uparrow \supset I^u
\]
1 The Rules

Recall that left rules correspond to “upside down elimination rules” and that right rules correspond to introduction rules.

\[
\begin{align*}
\frac{\Gamma, A \land B, A \Rightarrow C}{\Gamma, A \land B \Rightarrow C} & \quad \land L_1 \\
\frac{\Gamma, A \land B, B \Rightarrow C}{\Gamma \Rightarrow A \land B} & \quad \land R \\
\frac{\Gamma, A \lor B, A \Rightarrow C}{\Gamma, A \lor B \Rightarrow C} & \quad \lor L \\
\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \lor B} & \quad \lor R_1 \\
\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \lor B} & \quad \lor R_2 \\
\frac{\Gamma, A \supset B \Rightarrow A}{\Gamma, A \supset B \Rightarrow C} & \quad \supset L \\
\frac{\Gamma, A \supset B, B \Rightarrow C}{\Gamma \Rightarrow A \supset B} & \quad \supset R \\
\frac{\Gamma \Rightarrow \top}{\top \Rightarrow \top} & \quad \top \land L \\
\frac{\Gamma, \bot \Rightarrow C}{\bot \Rightarrow \top} & \quad \bot \lor R \\
\end{align*}
\]

2 Some Example Proofs

Task 1. \( \cdot \Rightarrow A \supset A \)

Solution 1:

\[
\frac{A \Rightarrow A}{\cdot \Rightarrow A \supset A} \supset R
\]

Task 2. \( \cdot \Rightarrow A \land B \supset B \land A \)

Solution 2:

\[
\begin{align*}
\frac{A \land B, B \Rightarrow B}{A \land B \Rightarrow B} & \quad \land L_2 \\
\frac{A \land B, A \Rightarrow A}{A \land B \Rightarrow A} & \quad \land R \\
\frac{A \land B \Rightarrow B \land A}{\cdot \Rightarrow A \land B \supset B \land A} & \quad \supset R
\end{align*}
\]

Task 3. \( \cdot \Rightarrow (A \supset (B \land C)) \supset (A \supset B) \)

Solution 3:

\[
\begin{align*}
\frac{(A \supset (B \land C)), A \Rightarrow A}{A \supset (B \land C) \Rightarrow A} & \quad \supset L_1 \\
\frac{(A \supset (B \land C)), A \Rightarrow B}{A \supset (B \land C) \Rightarrow B} & \quad \supset R
\end{align*}
\]

Task 4. \( \cdot \Rightarrow (A \supset B \supset C) \supset B \supset A \supset C \)
Solution 4:

\[
\begin{align*}
A \supset B \supset C, A & \Rightarrow A \\
A \supset B \supset C, B, A, B \supset C & \Rightarrow B \\
A \supset B \supset C, B, A, B \supset C & \Rightarrow C \\
\end{align*}
\]

Task 5. \( \cdot \Rightarrow (A \supset B) \supset ((A \land C) \supset (B \land C)) \)

Solution 5:

\[
\begin{align*}
(A \supset B), (A \land C), A & \Rightarrow A \\
(A \supset B), (A \land C) & \Rightarrow A \\
(A \supset B), (A \land C) & \Rightarrow B \\
(A \supset B), (A \land C) & \Rightarrow C \\
(A \supset B) & \Rightarrow ((A \land C) \supset (B \land C)) \\
\end{align*}
\]

3 Cuts

As a reminder, the cut theorem is as follows: If \( \Gamma \Rightarrow A \) and \( \Gamma, A \Rightarrow C \), then \( \Gamma \Rightarrow C \), where \( A \) and \( C \) are arbitrary propositions.

In class, we saw portions of the proof of admissibility for the cut rule.

Task 6. Finish the case for the proof of admissibility of cut where \( \mathcal{E} \) ends in \( \supset R \), and \( A \) is not the principal formula of the last inference in \( \mathcal{E} \).

Solution 6: We have that

\( \mathcal{D} = \Gamma \Rightarrow A \)

and

\( \mathcal{E} = \frac{E_1}{\Gamma, A, C_1 \Rightarrow C_2 \supset R} \)

\( C = C_1 \supset C_2 \)

this case

\( \Gamma, C_1 \Rightarrow A \)

weakening of \( \mathcal{D} \)

\( \Gamma, C_1 \Rightarrow C_2 \)

IH on \( A \), weakening of \( \mathcal{D} \), and \( \mathcal{E}_1 \)

\( \Gamma \Rightarrow C_1 \supset C_2 \)

by rule \( \supset R \) on above

Task 7. What would the derivations \( \mathcal{D} \) and \( \mathcal{E} \) look like if we wanted to do the same case as above, but with \( \supset L \) instead of \( \supset R \) as the last derivation in \( \mathcal{E} \)?

Solution 7:

\( \mathcal{D} = \Gamma \Rightarrow A \)

\( \mathcal{E} = \frac{E_1}{\Gamma', B_1 \supset B_2, A \Rightarrow B_1} \frac{E_2}{\Gamma', B_1 \supset B_2, A, B_2 \Rightarrow C \Rightarrow L} \)