ConstLog: Constructive Logic

Lecture Notes on Focusing

Frank Pfenning   André Platzer

Carnegie Mellon University || Karlsruhe Institute of Technology

Lecture 14

1 Introduction

In this lecture we will finally put much of what we have learned on proof theory together, following the slogan *focusing = inversion + chaining*. Focusing has been developed by Andreoli [And92] using classical linear logic, but it has proved to be a remarkably robust concept (see, for example, Liang and Miller [LM09]). We will follow the formulation of Simmons [Sim14], which includes particularly elegant proofs of the completeness of focusing using structural inductions.

2 The Need for Focus

Antecedents are helpful in a sequent calculus proof, because they represent everything one knows at the current state of proof search. At the same time, making use of assumptions is sometimes harder than it sounds. Making use of implications requires a small leap of faith with the hope that that implication is useful at all, which it will only be if we will be able to prove its assumption and then make use of the result. To illustrate, consider a simple example for atomic propositions a, b, c, d, and e.

$$a \wedge (a \supset d) \wedge (a \supset (b \vee c)) \wedge (b \supset c) \wedge (a \supset (c \vee d)) \wedge (e \supset c) \supset c$$

Once transformed by $\supset R$ and $\wedge L$, the resulting sequent clearly benefits from a use of $\supset L$:

$$a, a \supset d, a \supset (b \vee c), b \supset c, a \supset (c \vee d), e \supset c \longrightarrow c$$

But this begs the question, which implication of the antecedent to apply $\supset L$ to. Any wrong decision will have to be undone, which is particularly expensive if all orders of all decisions will have to be considered. The first implication $a \supset d$ will have a provable assumption but its outcome d is not ultimately helpful to establish the desired succedent c. The last implication $e \supset c$ has a useful outcome c, but no provable assumption e, so using $\supset L$ on it would result in an unprovable premise. The second implication $a \supset (b \vee c)$ is not alone useful, because it merely establishes $b \vee c$, not the desired c. The third implication $b \supset c$ also has a useful outcome c and no directly provable assumption b, but it has the benefit over the last implication of having an assumption that will be provable on the relevant parts of the proof that a use of the second implication retains. The fourth implication a$\supset(c \vee d)$ is similar to the second but its resulting case for disjunct d will not succeed.

Overall, this gives a lot of choices just from implications alone without even considering other connectives. If only there was a way of bringing more structure into this search question. The idea behind focusing is to follow through all the consequences of a decision in proof search without distraction by other propositions until it becomes clear whether it has a helpful outcome. We will put one implication in focus and keep on decomposing it until we see whether it helps.

3 Polarization

A key idea behind focusing is to limit nondeterminism by sequencing inferences on connectives that have similar behaviors. One behavior is that of *inversion*, perhaps slightly misnamed. Andreoli calls such connectives *asynchronous*, which expresses that when we see such a connective we can always decompose it. *Synchronous* connectives, by contrast, are those that "may have to wait" until they can be decomposed, but once we have committed to one by *focusing* on it, we can continue to chain inferences on this one proposition and don't need to look elsewhere.

If we classify propositions by their behavior as *succedents*, then so-called *negative propositions* are asynchronous or, to say it differently, have invertible right rules. Conversely, *positive propositions* are asynchronous when they appear as *antecedents*, or, to say it differently, have invertible left rules. The so-called *shift operators* go back and forth between positive and negative propositions so that any proposition can be polarized differently.

$$
\begin{array}{lrcl}
\text{Neg. Props.} & A^-, B^- & ::= & A^+ \supset B^- \mid A^- \wedge B^- \mid \top \mid P^- \mid {\uparrow}A^+ \\
\text{Pos. Props.} & A^+, B^+ & ::= & A^+ \vee B^+ \mid \bot \mid A^+ \wedge B^+ \mid \top \mid P^+ \mid {\downarrow}A^-
\end{array}
$$

Implication expects different polarities left and right, because the $\supset R$ rule moves the assumption into the antecedent. A few notes:

Conjunction and truth: Conjunction $A \wedge B$ and truth $\top$ appear as both positive and negative propositions. That's because there are invertible rules for conjunction both in the antecedent and the succedent. Really, it should be seen as an indication that there are two different conjunctions $A^- \wedge^- B^-$ and $A^+ \wedge B^+$ and two different truth constants $\top^-$ and $\top^+$ with different rules that happen to be *logically equivalent* even though they have different intrinsic properties, both from the perspective of proof search and the computational contents of proofs. For example, in a functional language, positive conjunction would correspond to an eager pairs, while negative conjunction corresponds to lazy pairs.

So, if we take proofs seriously as defining the meaning of propositions there should be two conjunctions, which are disambiguated in the polarized presentation of logic.

Atoms: Atoms may be viewed from one perspective as propositional variables, from another as "uninterpreted" propositions which means that only the logical assumptions we make about them imbue them with meaning, and from a third as placeholders for other propositions (that may be either positive or negative). Each can be independently assigned an arbitrary polarity, as long as all occurrences of an atom are given the same polarity.

Quantifiers: The universal quantifier is negative since its right rule is invertible, while the existential quantifier is positive since its left rule is. We do not treat them formally to avoid the syntactic complication of introducing terms, parameters, their types, and the relevant typing judgments.

Shifts: The shift operators embed operators of the wrong polarity indicating explicit boundaries where proof search switches from right to left or postpones deterministic decompositions in the antecedent. Shifts will syntactically mark transition points between inversion and chaining phases in focused proof search.

4 Inversion

Inversion decomposes all asynchronous connectives until we reach a sequent where all proposition in the sequent are either atoms or synchronous. In order for inversion to proceed deterministically, the proof calculus first decomposes asynchronous connectives in the succedent and then in the antecedent. We use an *ordered context* Ω^+ (as in Lecture 12) consisting of all positive propositions of the antecedent, leaving context Γ for stable antecedents. Positive propositions are stable succedents, negative propositions are stable antecedents, and atomic propositions are stable anywhere, because they cannot themselves be decomposed.

$$
\begin{array}{lll}
\text{Stable succedent} & \mathsf{e} & ::= \ A^+ \mid P^- \\
\text{Stable antecedents} & \Gamma & ::= \ \cdot \mid \Gamma, A^- \mid \Gamma, P^+ \\[1ex]
\text{Right inversion} & \Gamma \,;\, \Omega^+ \xrightarrow{R} A^- & \\
\text{Left inversion} & \Gamma \,;\, \Omega^+ \xrightarrow{L} \mathsf{e} & \\
\text{Stable sequent} & \Gamma \longrightarrow \mathsf{e} &
\end{array}
$$

The inversion rules are summarized in Figure 1. Similar to the pure inversion calculus from Lecture 12, these rules first decompose asynchronous connectives in the succedent (so negative connectives with invertible right rules) on the right, then asynchronous

$$\frac{\Gamma\,;\,\Omega^+ \cdot A^+ \xrightarrow{R} B^-}{\Gamma\,;\,\Omega^+ \xrightarrow{R} A^+ \supset B^-}\supset\!R$$

$$\frac{\Gamma\,;\,\Omega^+ \xrightarrow{R} A^- \quad \Gamma\,;\,\Omega^+ \xrightarrow{R} B^-}{\Gamma\,;\,\Omega^+ \xrightarrow{R} A^- \wedge B^-}\wedge R \qquad \frac{}{\Gamma\,;\,\Omega^+ \xrightarrow{R} \top}\top R$$

$$\frac{\Gamma\,;\,\Omega^+ \xrightarrow{L} P^-}{\Gamma\,;\,\Omega^+ \xrightarrow{R} P^-}pR \qquad \frac{\Gamma\,;\,\Omega^+ \xrightarrow{L} A^+}{\Gamma\,;\,\Omega^+ \xrightarrow{R} {\uparrow}A^+}{\uparrow}R$$

$$\frac{\Gamma\,;\,\Omega^+ \cdot A^+ \xrightarrow{L} \mathsf{e} \quad \Gamma\,;\,\Omega^+ \cdot B^+ \xrightarrow{L} \mathsf{e}}{\Gamma\,;\,\Omega^+ \cdot A^+ \vee B^+ \xrightarrow{L} \mathsf{e}}\vee L \qquad \frac{}{\Gamma\,;\,\Omega^+ \cdot \bot \xrightarrow{L} \mathsf{e}}\bot L$$

$$\frac{\Gamma\,;\,\Omega^+ \cdot A^+ \cdot B^+ \xrightarrow{L} \mathsf{e}}{\Gamma\,;\,\Omega^+ \cdot A^+ \wedge B^+ \xrightarrow{L} \mathsf{e}}\wedge L \qquad \frac{\Gamma\,;\,\Omega^+ \xrightarrow{L} \mathsf{e}}{\Gamma\,;\,\Omega^+ \cdot \top \xrightarrow{L} \mathsf{e}}\top L$$

$$\frac{\Gamma, P^+\,;\,\Omega^+ \xrightarrow{L} \mathsf{e}}{\Gamma\,;\,\Omega^+ \cdot P^+ \xrightarrow{L} \mathsf{e}}pL \qquad \frac{\Gamma, A^-\,;\,\Omega^+ \xrightarrow{L} \mathsf{e}}{\Gamma\,;\,\Omega^+ \cdot {\downarrow}A^- \xrightarrow{L} \mathsf{e}}{\downarrow}L$$

$$\frac{\Gamma \longrightarrow \mathsf{e}}{\Gamma\,;\,\cdot \xrightarrow{L} \mathsf{e}}\mathsf{stable}$$

Figure 1: Inversion phase of focusing

connectives in the antecedents (so positive connectives with invertible left rules), until they reach stable sequents with purely negative antecedents or atomic propositions and a positive succedent or atomic proposition. Stable sequents crystallize the situation where choices are required to continue proof search. Compared to the unstructured search that begins in the pure inversion calculus from Lecture 12 after completion of inversion, the focusing calculus will structure the search through the respective proof rules for asynchronous connectives.

5 Chaining

Once inversion has completed, so all asynchronous connectives have been decomposed deterministically (invertible rules), we have to focus on a single proposition, either a positive succedent or a negative antecedent, and then chain together all inferences on that one proposition in focus. In particular, *no other propositions are considered*, and only

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow A^+}\ \mathsf{focus}R \qquad \frac{\Gamma, A^-, [A^-] \longrightarrow \mathsf{e}}{\Gamma, A^- \longrightarrow \mathsf{e}}\ \mathsf{focus}L$$

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee R_1 \qquad \frac{\Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee R_2 \qquad \text{no right rule for } [\bot]$$

$$\frac{\Gamma \longrightarrow [A^+] \quad \Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \wedge B^+]}\ \wedge R \qquad \frac{}{\Gamma \longrightarrow [\top]}\ \top R$$

$$\frac{}{\Gamma, P^+ \longrightarrow [P^+]}\ \mathsf{id}^+ \qquad \frac{\Gamma\ ;\ \cdot \xrightarrow{R} A^-}{\Gamma \longrightarrow [\downarrow A^-]}\ \downarrow R$$

$$\frac{\Gamma \longrightarrow [A^+] \quad \Gamma, [B^-] \longrightarrow \mathsf{e}}{\Gamma, [A^+ \supset B^-] \longrightarrow \mathsf{e}}\ \supset L$$

$$\frac{\Gamma, [A^-] \longrightarrow \mathsf{e}}{\Gamma, [A^- \wedge B^-] \longrightarrow \mathsf{e}}\ \wedge L_1 \qquad \frac{\Gamma, [B^-] \longrightarrow \mathsf{e}}{\Gamma, [A^- \wedge B^-] \longrightarrow \mathsf{e}}\ \wedge L_2 \qquad \text{no left rule for } [\top]$$

$$\frac{}{\Gamma, [P^-] \longrightarrow P^-}\ \mathsf{id}^- \qquad \frac{\Gamma\ ;\ A^+ \xrightarrow{L} \mathsf{e}}{\Gamma, [\uparrow A^+] \longrightarrow \mathsf{e}}\ \uparrow L$$

Figure 2: Chaining phase of focusing

one proposition can be in focus in any sequent. The idea is that once we focus on a proposition we continue to apply rules to its rules until we can determine whether that proof succeeds (or loses focus at an explicit shift). The focus on a proposition A will be written $[A]$. This gives us two new forms of judgments.

Right focus	$\Gamma \longrightarrow [A^+]$
Left focus	$\Gamma, [A^-] \longrightarrow \mathsf{e}$

The chaining rules that chain together reasoning for the one proposition in focus (and all that stems from it) can be found in Figure 2. Some remarks:

Atoms: Much of the power of focusing comes from the fact that left focus $[P^-]$ *fails* unless the succedent is also P^-. Dually, right focus $[P^+]$ *fails* unless P^+ is one of the antecedents. It is not possible to focus on a negative atom in the succedent or a positive atom in the antecedent, because those polarities indicate asynchronous cases part of the inversion phase, except for the fact that atomic propositions cannot be decomposed.

Shifts: In contrast, the shifting rules $\uparrow L$ and $\downarrow R$ just lose focus and return to the appropriate inversion judgment, since they carry a proposition of the opposite polarity.

The $\supset L$ rule, for instance, requires that an antecedent implication $A^+ \supset B^-$ be in focus before using it. Once that commitment has been made, the rule continues to focus on proving its assumption A^+ in the left premise and focuses on making use of the resulting B^- in the right premise. If either of the two chained-together inferences fail, putting the focus on using that implication was the wrong decision and will be undone in proof search. This focus avoids interleaving decisions that require commitment with the use of other proof rules that do not result from that decision. The $\supset L$ rule may first suspiciously lose the implication $A^+ \supset B^-$ just like $\wedge L_i$ may suspiciously misplace the conjunction. But the $\mathsf{focus}L$ rule retains an unfocused version of the focused negative antecedent for precisely that purpose. Indeed, the $\wedge L_1$ rule could hardly keep both A^- and B^- in focus, because any focused sequent must have exactly one formula in focus. Since a focus is also a commitment to use the proposition in focus for the proof (as becomes ultimately clear in rules id^+ and id^-), it would also be devastating if $\wedge L_1$ were to force the use of both parts of a conjunction.

6 Flipping Polarities

Consider a simple transitivity example:

$$a, a \supset b, b \supset c \longrightarrow c$$

resulting from the following by inversion with $\supset R$ and $\wedge L$:

$$\longrightarrow (a \wedge (a \supset b) \wedge (b \supset c)) \supset c$$

Let us polarize all atoms negatively and add the minimal number of shifts:

$$a^-, {\downarrow} a^- \supset b^-, {\downarrow} b^- \supset c^- \longrightarrow c^-$$

where the (unary) shift operators binds tightest.

Without focus there are many different proofs. In particular, we could apply $\supset L$ to $a \supset b$, or to $b \supset c$. With focus, there is only one possible proof: at each choice point, when we focus, only on possibility will succeed and the others will fail immediately. You should convince yourself that this is the case. We elide stable and define $\Gamma_0 = (a^-, {\downarrow} a^- \supset b^-, {\downarrow} b^- \supset c^-)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_0, [a^-] \longrightarrow a^-}\,\mathsf{id}^-}{\Gamma_0 \longrightarrow a^-}\,\mathsf{focus}L}{\Gamma_0 \longrightarrow [{\downarrow} a^-]}\,{\downarrow}R \qquad \dfrac{}{\Gamma_0, [b^-] \longrightarrow b^-}\,\mathsf{id}^-}{\Gamma_0, [{\downarrow} a^- \supset b^-] \longrightarrow b^-}\,{\supset}L}{\Gamma_0 \longrightarrow b^-}\,\mathsf{focus}L}{\Gamma_0 \longrightarrow [{\downarrow} b^-]}\,{\downarrow}R \qquad \dfrac{}{\Gamma_0, [c^-] \longrightarrow c^-}\,\mathsf{id}^-}{\dfrac{\Gamma_0, [{\downarrow} b^- \supset c^-] \longrightarrow c^-}{\Gamma_0 \longrightarrow c^-}\,\mathsf{focus}L}\,{\supset}L
$$

Let's go back to our sequent

$$a, a \supset b, b \supset c \longrightarrow c$$

Now polarize all the atoms positively and add minimal number of shifts:

$$a^+, a^+ \supset {\uparrow} b^+, b^+ \supset {\uparrow} c^+ \longrightarrow c^+$$

Now we must focus on $a^+ \supset {\uparrow} b^+$ first, then $b^+ \supset {\uparrow} c^+$, then c^+. All other attempts at focusing will either fail, or conclude a fact that is already in the database. We abbreviate $\Gamma_1 = (a^+, a^+ \supset {\uparrow} b^+, b^+ \supset {\uparrow} c^+)$

$$
\dfrac{\dfrac{a^+ \in \Gamma_1}{\Gamma_1 \longrightarrow [a^+]}\,\mathsf{id}^+ \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1, b^+ \longrightarrow [b^+]}\,\mathsf{id}^+ \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_1, b^+, c^+ \longrightarrow [c^+]}\,\mathsf{id}^+}{\Gamma_1, b^+, c^+ \longrightarrow c^+}\,\mathsf{focus}R}{\Gamma_1, b^+, [{\uparrow} c^+] \longrightarrow c^+}\,{\uparrow}L}{\Gamma_1, b^+, [b^+ \supset {\uparrow} c^+] \longrightarrow c^+}\,{\supset}L}{\dfrac{\Gamma_1, b^+ \longrightarrow c^+}{\Gamma_1, [{\uparrow} b^+] \longrightarrow c^+}\,{\uparrow}L}\,\mathsf{focus}L}{}}{\dfrac{\Gamma_1, [a^+ \supset {\uparrow} b^+] \longrightarrow c^+}{\Gamma_1 \longrightarrow c^+}\,\mathsf{focus}L}\,{\supset}L
$$

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[LM09] Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, November 2009.

[Sim14] Robert J. Simmons. Structural focalization. *Transactions on Computational Logic*, 15(3):21:1–21:33, July 2014.