Constructive Logic (15-317), Spring 2021
Recitation 9: Midterm 2 Review (3-31-2021)
adapted from Fall 2015 handout (by Vincent Huang, Anna Gommerstadt, and Michael Coblenz)

1 Midterm Logistics

The midterm will be held the upcoming Tuesday (April 6) during lecture. It will be open book, but actions such as browsing the internet or interacting with others are prohibited.

2 Sequent calculus

The following tasks all use the original, non-restricted sequent calculus.

Task 1. What's wrong with this proof?

$$\dfrac{\overline{(A \lor B) \land C, A \lor B \implies A \lor B}\ \text{init}}{(A \lor B) \land C \implies A \lor B}\ \land L_1$$

Solution 1: The *init* rule cannot be applied on non-atomic propositions like it is in the proof.

Task 2. How about this one?

$$\dfrac{\overline{A \implies A}\ \text{init} \qquad \dfrac{\overline{A, B \implies B}\ \text{init}}{A \implies B}\ \text{weak}}{A \implies A \land B}\ \land R$$

Solution 2: When going upwards on a proof tree, weakening should only be used to drop antecedents. It does not allow one to add new antecedents on sequents that are not proven yet.

2.1 Unprovable stuff

Task 3. Show that $\neg(A \supset B) \supset (A \land \neg B)$ is not provable in sequent calculus.

Task 4. Show that $\neg(\neg A \land \neg B) \supset (A \lor B)$ is not provable in sequent calculus.

3 Cut elimination

- $\mathcal{D}$ is the proof of $\Gamma \implies A$.
- $\mathcal{E}$ is the proof of $\Gamma, A \implies C$.
- Cut is the theorem that if $\Gamma \implies A$ and $\Gamma, A \implies C$, then $\Gamma \implies C$.
- Cut elimination is proven by lexicographic induction over A, then $\mathcal{D}$, then $\mathcal{E}$.
- When we appeal to the IH in this proof, "smaller" means contains fewer connectives (when we say A grows smaller), or contains fewer proof steps (when we say $\mathcal{D}$ or $\mathcal{E}$ grows smaller).
- The "principal formula" is the thing that the last rule applied in the derivation operated on, i.e. decomposed or whatever.

4 Restricted sequent calculus

What to remember:

- the motivation behind reduced sequent calculus rules.
- the idea of reducing the "weight" when going bottom-up.
- how the proofs for soundness and completeness in relation to the original sequent calculus works

Task 5. Suppose we replaced the $\wedge \supset L$ rule with

$$\frac{\Gamma, A_1 \supset B, A_2 \supset B \longrightarrow C}{\Gamma, (A_1 \wedge A_2) \supset B \longrightarrow C} \; Rep$$

Is this rule sound and complete with respect to the usual restricted sequent calculus? Are there any consequences of this replacement for the termination of the proof search decision procedure?

Solution 5: The rule is not sound. Many propositions provable by the *Rep* rule are not provable with the original $\wedge \supset L$ rule. The *Rep* rule lets us prove $A, (A \wedge \bot) \supset B \longrightarrow B$ with *Rep* and $P \supset L$. However, this clearly should not work.

The rule is complete. Any proposition provable by the $\wedge \supset L$ rule is provable by *Rep* because having $A_1 \supset B$ and $A_2 \supset B$ is stronger than just having $A_1 \supset A_2 \supset B$, since a proof does not have to deal with A_1 before using $A_2 \supset B$.

5 Inversion

We've discussed invertibility in the past, but to recap, how do we know if a rule is invertible?

A rule is invertible when the premises hold iff the conclusion holds. One way to show this is, given the conclusion, producing a derivation that reconstructs its premises. So,

- To show invertibility, start with the conclusion of the relevant rule, and apply other rules to get to the premises. As a simple example, to show that $\wedge I$ is invertible, $\dfrac{A \wedge B \; true}{A \; true} \wedge E_1$ $\dfrac{A \wedge B \; true}{B \; true} \wedge E_2$
  so, since (by using other rules in the system) we can get back both premises from the conclusion, it's invertible.
- To show non-invertibility, simply exhibit a counterexample! There's no need for convoluted arguments. For instance, to show that the $\vee I$ rules are not invertible, take $A = \top$ and $B = \bot$. Clearly $A \vee B \; true$ holds, but one can't conclude from that that $B \; true$ holds (and it doesn't). So $\vee I_2$ is non-invertible, and the same argument works for $\vee I_1$.

Proofs of invertibility may look somewhat reminiscent of local expansions, because you generally have a connective, and then you apply elimination rules to "get rid" of the connective and get back the connective-free premises. But

- A local completeness proof would involve reintroducing the connective after this, which is wholly unnecessary in this case!
- They only seem to be similar because all the things you've been trying to show are invertible have been introduction rules. This doesn't always have to be the case.

Another important thing to note is that invertibility depends heavily on the *presentation* of the rule. For instance, sequent calculus as presented in this class was entirely motivated by a correspondence to verifications and uses, and thereby natural deduction. Yet $\wedge E_L$ and $\wedge E_R$ are not invertible, but all the $\wedge L$ rules are invertible.

When showing invertibility for sequent calculus rules, you may use theorems such as cut and weakening. You are not constrained to just the rules of the sequent calculus.

6 Focusing

When doing a proof with the focusing rules presented in Lecture 14,

- In the inversion phase, right rules can only be applied when the succedent is negative.
- Similarly, left rules can only be applied when an antecedent is positive.
- Right rules are applied before left rules in the inversion phase.
- The $focusR$ and $focusL$ rules transition a proof from the inversion phase to the chaining phase.
- The $R\downarrow$ and $L\uparrow$ rules transition a proof from the chaining phase back to the inversion phase. They result in a loss of focus on the current proposition.

Task 6. Pick a polarization of the atoms in $((A \vee B) \supset C) \wedge A \supset C$ and prove it with the focusing rules.

Solution 6: We pick A^+, B^-, and C^-, which gives us $\downarrow ((A^+ \vee \downarrow B^-) \supset C^-) \wedge A^+ \supset C^-$. We let $\Gamma_0 = (A^+ \vee \downarrow B^-) \supset C^-, A^+$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_0 \longrightarrow [A^+]}\,id{+}}{\Gamma_0 \longrightarrow [A^+ \vee \downarrow B^-]}\,\vee R_1 \qquad \dfrac{}{\Gamma_0, [C^-] \longrightarrow C^-}\,id{-}}{\Gamma_0, [(A^+ \vee \downarrow B^-) \supset C^-] \longrightarrow C^-}\,\supset L}{\Gamma_0 \longrightarrow C^-}\,focusL}{(A^+ \vee \downarrow B^-) \supset C^-, A^+; \cdot \longrightarrow^L C^-}\,stable}{(A^+ \vee \downarrow B^-) \supset C^-; A^+ \longrightarrow^L C^-}\,pL}{;\downarrow ((A^+ \vee B^-) \supset C^-), A^+ \longrightarrow^L C^-}\,\downarrow L}{;\downarrow ((A^+ \vee B^-) \supset C^-) \wedge A^+ \longrightarrow^L C^-}\,\wedge L}{;\downarrow ((A^+ \vee B^-) \supset C^-) \wedge A^+ \longrightarrow^R C^-}\,pR}{\longrightarrow^R \downarrow ((A^+ \vee \downarrow B^-) \supset C^-) \wedge A^+ \supset C^-}\,\supset R
$$