Constructive Logic (15-317), Spring 2021
Recitation 7: Sequent Calculus and Cut Elimination (2021-03-17)
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1 The Rules

Recall that left rules correspond to "upside down elimination rules" and that right rules correspond to introduction rules.

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L_2 \qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L \qquad \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2$$

$$\text{No } \top L. \qquad \frac{}{\Gamma \Longrightarrow \top}\ \top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ \bot L \qquad \text{No } \bot R.$$

$$\frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L \qquad \frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R$$

$$\frac{}{\Gamma, A \Longrightarrow A}\ \mathsf{id}$$

2 Some Example Proofs

Task 1. $\cdot \Longrightarrow A \supset A$

Solution 1:

$$\dfrac{\dfrac{}{A \Longrightarrow A}\ \mathsf{id}}{\cdot \Longrightarrow A \supset A}\ \supset R$$

Task 2. $\cdot \Longrightarrow A \wedge B \supset B \wedge A$

Solution 2:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B, B \Longrightarrow B}\ \mathsf{id}}{A \wedge B \Longrightarrow B}\ \wedge L_2 \quad \dfrac{\dfrac{}{A \wedge B, A \Longrightarrow A}\ \mathsf{id}}{A \wedge B \Longrightarrow A}\ \wedge L_1}{A \wedge B \Longrightarrow B \wedge A}\ \wedge R}{\cdot \Longrightarrow A \wedge B \supset B \wedge A}\ \supset R$$

Task 3. $\cdot \Longrightarrow (A \supset (B \wedge C)) \supset (A \supset B)$

Solution 3:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset (B \wedge C)), A \Longrightarrow A}\ \mathsf{id} \quad \dfrac{\dfrac{}{(A \supset (B \wedge C)), A, B \wedge C, B \Longrightarrow B}\ \mathsf{id}}{(A \supset (B \wedge C)), A, B \wedge C \Longrightarrow B}\ \wedge L_1}{(A \supset (B \wedge C)), A \Longrightarrow B}\ \supset L}{(A \supset (B \wedge C)) \Longrightarrow (A \supset B)}\ \supset R}{\cdot \Longrightarrow (A \supset (B \wedge C)) \supset (A \supset B)}\ \supset R$$

Task 4. $\cdot \Longrightarrow (A \supset B \supset C) \supset B \supset A \supset C$

Solution 4:

$$
\dfrac{\dfrac{}{A \supset B \supset C, B, A \Longrightarrow A}\;\mathsf{id} \qquad \dfrac{\dfrac{}{A \supset B \supset C, B, A, B \supset C \Longrightarrow B}\;\mathsf{id} \quad \dfrac{}{A \supset B \supset C, B, A, B \supset C, C \Longrightarrow C}\;\mathsf{id}}{A \supset B \supset C, B, A, B \supset C \Longrightarrow C}\;{\supset}L}{\dfrac{A \supset B \supset C, B, A \Longrightarrow C}{\dfrac{A \supset B \supset C, B \Longrightarrow A \supset C}{\dfrac{A \supset B \supset C \Longrightarrow B \supset A \supset C}{\cdot \Longrightarrow (A \supset B \supset C) \supset B \supset A \supset C}\;{\supset}R}\;{\supset}R}\;{\supset}R}\;{\supset}L
$$

Task 5. $\cdot \Longrightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))$

Solution 5:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset B), (A \wedge C), A \Longrightarrow A}\;\mathsf{id}}{(A \supset B), (A \wedge C) \Longrightarrow A}\;{\wedge}L_1 \quad \dfrac{}{(A \supset B), (A \wedge C), B \Longrightarrow B}\;\mathsf{id}}{(A \supset B), (A \wedge C) \Longrightarrow B}\;{\supset}L \qquad \dfrac{\dfrac{}{(A \supset B), (A \wedge C), C \Longrightarrow C}\;\mathsf{id}}{(A \supset B), (A \wedge C) \Longrightarrow C}\;{\wedge}L_2}{\dfrac{(A \supset B), (A \wedge C) \Longrightarrow B \wedge C}{\dfrac{(A \supset B) \Longrightarrow ((A \wedge C) \supset (B \wedge C))}{\cdot \Longrightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))}\;{\supset}R}\;{\supset}R}\;{\wedge}R
$$

3 Cuts

As a reminder, the cut theorem is as follows: If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$, then $\Gamma \Longrightarrow C$, where A and C are arbitrary propositions.

In class, we saw portions of the proof of admissibility for the cut rule.

Task 6. Finish the case for the proof of admissibility of cut where $\mathcal{E}$ ends in $\supset R$, and A is not the principal formula of the last inference in $\mathcal{E}$.

Solution 6: We have that

$$\mathcal{D} = \Gamma \Longrightarrow A$$

and

$$\mathcal{E} = \quad \dfrac{\overset{E_1}{\Gamma, A, C_1 \Longrightarrow C_2}}{\Gamma, A \Longrightarrow C_1 \supset C_2}\;{\supset}R$$

$C = C_1 \supset C_2$	this case
$\Gamma, C_1 \Longrightarrow A$	weakening of $\mathcal{D}$
$\Gamma, C_1 \Longrightarrow C_2$	IH on A, weakening of $\mathcal{D}$, and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_1 \supset C_2$	by rule $\supset R$ on above

Task 7. What would the derivations $\mathcal{D}$ and $\mathcal{E}$ look like if we wanted to do the same case as above, but with $\supset L$ instead of $\supset R$ as the last derivation in $\mathcal{E}$?

Solution 7:

$$\mathcal{D} = \Gamma \Longrightarrow A$$

$$\mathcal{E} = \quad \dfrac{\overset{E_1}{\Gamma', B_1 \supset B_2, A \Longrightarrow B_1} \quad \overset{E_2}{\Gamma', B_1 \supset B_2, A, B_2 \Longrightarrow C}}{\Gamma', B_1 \supset B_2, A \Longrightarrow C}\;{\supset}L$$