Constructive Logic (15-317), Spring 2021
Recitation 6: Judgmental Natural Deduction (2021-03-10)
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1 Turnstile Notation

Our natural deduction rules can also be written in a form involving turnstiles and contexts. Instead of having hypotheses be represented vertically above the conclusions they can be used to derive, they are simply put inside the context, which we represent in our rules with the symbol Γ. The context is simply an unordered list of hypotheses. We may refer to these hypotheses as antecedents.

A hypothetical judgment has the form $\Gamma \vdash A$. The judgment on the right side of the turnstile is often referred to as the succedent of the hypothetical judgment.

All of our existing natural deduction rules have a corresponding representation with turnstile notation, as shown below.

$$\frac{\Gamma \vdash A true \quad \Gamma \vdash B true}{\Gamma \vdash A \wedge B true} \wedge I \qquad \frac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash A\ true} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash B\ true} \wedge E_2$$

$$\frac{\Gamma, A\ true \vdash B\ true}{\Gamma \vdash A \supset B\ true} \supset I \qquad \frac{\Gamma \vdash A \supset B\ true \quad \Gamma \vdash A\ true}{\Gamma \vdash B\ true} \supset E$$

$$\frac{\Gamma \vdash A\ true}{\Gamma \vdash A \vee B\ true} \vee I_1 \qquad \frac{\Gamma \vdash B\ true}{\Gamma \vdash A \vee B\ true} \vee I_2 \qquad \frac{\Gamma \vdash A \vee B\ true \quad \Gamma, A\ true \vdash C\ true \quad \Gamma, B\ true \vdash C\ true}{\Gamma \vdash C\ true} \vee E$$

$$\frac{}{\Gamma \vdash \top\ true} \top I \qquad \frac{\Gamma \vdash \bot\ true}{\Gamma \vdash C\ true} \bot E$$

There is one more rule that exists when using the turnstile notation, however. Without it, we are not able to complete most proofs. For example, if using this notation to prove $A \wedge B \supset A$:

$$\cfrac{\cfrac{A \wedge B\ true \vdash A \wedge B\ true}{A \wedge B\ true \vdash A\ true} \wedge E_2}{\vdash A \wedge B \supset A\ true} \supset I$$

Task 1. The judgment $A \wedge B\ true \vdash A \wedge B\ true$ clearly makes sense, but none of the existing rules give us a way to justify this. What rule do we need to be able to finish off this proof, then?

Solution 1: We need a rule which allows us to conclude that $\Gamma \vdash J$, if $J \in \Gamma$. It looks like this:

$$\frac{J \in \Gamma}{\Gamma, J\ true \vdash J\ true} hyp$$