Constructive Logic (15-317), Spring 2021
Recitation 4: Quantifiers (2021-02-24)
André Platzer et al

1 Quantifiers

Up to now, we have been vague about what, exactly, our atomic propositions A are representing. In order to discuss quantification, however, we need to be precise over what, exactly, we are quantifying over. We do this via a new judgment $t : \tau$, where τ is some to-be-defined type. Oftentimes, we are interested in some particular type, like the type of natural numbers or the type of Turing Machines, but the meaning of the $\exists$ and $\forall$ connectives are independent of this.
The rules for verifying these are as follows:

$$\dfrac{\begin{array}{c}\overline{a : A}\\ \vdots \\ A(a)\uparrow\end{array}}{\forall x : \tau.\, A(x)\uparrow}\ \forall\mathsf{I}^{a} \qquad\qquad \dfrac{\forall x : \tau.\, A(x)\downarrow \quad t : \tau}{A(t)\downarrow}\ \forall\mathsf{E}$$

$$\dfrac{t : \tau \quad A(t)\uparrow}{\exists x : \tau.\, A(x)\uparrow}\ \exists\mathsf{I} \qquad\qquad \dfrac{\exists x : \tau.\, A(x)\downarrow \quad \begin{array}{c}\overline{a : \tau}\quad \overline{A(a)\downarrow}^{\,u}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\ \exists\mathsf{E}^{a,u}$$

By now, you should be comfortable with erasing the arrows to recover the rules defining these connectives for natural deduction. The intuition for these rules should be straightforward – to prove that some proposition $A(x)$ is true for all $x : \tau$, we should be able to derive $A(c)$ *true* for some arbitrary $c : \tau$. Similarly, we can introduce an existential by demonstrating some object satisfying the proposition.

Eliminating foralls is similarly simple. To eliminate an existential, however, we must do a little more work. If we have $\exists x : \tau.\, A(x)$, then we may not assume anything else about the witness! It must be an object of type τ, and also that it satisfies $A(x)$, but any other properties must be abstracted out, to be replaced with an arbitrary object with the known properties.

2 Examples with quantifiers

Consider predicates $A(x)$ and $B(x)$ which depend on $x : \tau$.

Task 1. Show $\forall x : \tau.\, A(x) \wedge B(x) \supset \forall x : \tau.\, A(x) \wedge \forall x : \tau.\, B(x)$ *true.*

Solution 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x : \tau.\, A(x) \wedge B(x)\ \mathit{true}}^{\,p} \quad \overline{u : \tau}}{A(u) \wedge B(u)\ \mathit{true}}\ \forall\mathsf{E}}{A(u)\ \mathit{true}}\ \wedge\mathsf{E}_1}{\forall x : \tau.\, A(x)\ \mathit{true}}\ \forall\mathsf{I}_u \quad \dfrac{\dfrac{\dfrac{\overline{\forall x : \tau.\, A(x) \wedge B(x)\ \mathit{true}}^{\,p} \quad \overline{v : \tau}}{A(v) \wedge B(v)\ \mathit{true}}\ \forall\mathsf{E}}{A(v)\ \mathit{true}}\ \wedge\mathsf{E}_2}{\forall x : \tau.\, B(x)\ \mathit{true}}\ \forall\mathsf{I}_v}{\forall x : \tau.\, A(x) \wedge \forall x : \tau.\, B(x)\ \mathit{true}}\ \wedge\mathsf{I}}{\forall x : \tau.\, A(x) \wedge B(x) \supset \forall x : \tau.\, A(x) \wedge \forall x : \tau.\, B(x)\ \mathit{true}}\ {\supset}\mathsf{I}^{p}$$

Next, let $A(x, y)$ be a formula with two variables $x : \tau$ and $y : \sigma$.

Task 2. Show that you can "swap" an existential and universal. Do a verification proof.

Solution 2:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{u}
\UnaryInfC{$\exists y : \tau.\, \forall x : \sigma.\, A(x,y) \downarrow$}
\AxiomC{}
\UnaryInfC{$d : \tau$}
\AxiomC{}
\RightLabel{v}
\UnaryInfC{$\forall x : \sigma.\, A(x,d) \downarrow$}
\AxiomC{}
\UnaryInfC{$c : \sigma$}
\RightLabel{$\forall\mathsf{E}$}
\BinaryInfC{$A(c,d) \downarrow$}
\RightLabel{$\updownarrow$}
\UnaryInfC{$A(c,d) \uparrow$}
\RightLabel{$\exists\mathsf{I}$}
\BinaryInfC{$\exists y : \tau.\, A(c,y) \uparrow$}
\RightLabel{$\exists\mathsf{E}^{d,v}$}
\BinaryInfC{$\exists y : \tau.\, A(c,y) \uparrow$}
\RightLabel{$\forall\mathsf{I}^{c}$}
\UnaryInfC{$\forall x : \sigma.\, \exists y : \tau.\, A(x,y) \uparrow$}
\RightLabel{$\supset\mathsf{I}^{u}$}
\UnaryInfC{$(\exists y : \tau.\, \forall x : \sigma.\, A(x,y)) \supset (\forall x : \sigma.\, \exists y : \tau.\, A(x,y)) \uparrow$}
\end{prooftree}
$$

3 Classical Logic

Classical logic is not on the upcoming homework and will be taught later in the course, but since there is only one lecture of material to cover in this recitation, we have a preview.

There exists a judgment A *false* in classical logic which does not exist in constructive logic. The key component of classical logic is proof by contradiction, which puts the judgments A *true* and A *false* in opposition to one another. The same connectives exist in classical logic, but there are only introduction rules, since the negation operator allows for switching between the two modes of true and false.

The conjunction rules are as follows:

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}} \wedge T \qquad \frac{A\ \mathit{false}}{A \wedge B\ \mathit{false}} \wedge F_1 \qquad \frac{B\ \mathit{false}}{A \wedge B\ \mathit{false}} \wedge F_2$$

Task 3. Give the rules for disjunction, implication, and negation.

Solution 3: Disjunction:

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}} \vee T_1 \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}} \vee T_2 \qquad \frac{A\ \mathit{false} \quad B\ \mathit{false}}{A \vee B\ \mathit{false}} \vee F$$

Implication:

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}} \supset T^u \qquad \frac{A\ \mathit{true} \quad B\ \mathit{false}}{A \supset B\ \mathit{false}} \supset F$$

Negation:

$$\frac{A\ \mathit{false}}{\neg A\ \mathit{true}} \neg T \qquad \frac{A\ \mathit{true}}{\neg A\ \mathit{false}} \neg F$$

In this system, contradiction is represented with #.

Task 4. What are the rules for contradiction? How do they differ from the definition of ¬ and the not-elimination rule in constructive logic?

Solution 4:

$$\frac{A\ \mathit{true} \quad A\ \mathit{false}}{\#} \#I \qquad \frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ \#\end{array}}{A\ \mathit{false}} F\#^u \qquad \frac{\begin{array}{c}\overline{A\ \mathit{false}}^{\,u} \\ \vdots \\ \#\end{array}}{A\ \mathit{true}} T\#^u$$

The $F\#^u$ rule is very similar to the definition of ¬ in constructive logic. The derived rule for not elimination in constructive logic is very similar to the rule with # as its conclusion.