Constructive Logic (15-317), Spring 2021
Recitation 3: Harmonic proofs and verifications (2021-02-17)
André Platzer et al

1 Harmony

Proof-theoretic harmony is a necessary, but not sufficient, condition for the well-behavedness of a logic; harmony ensures that the connectives are *locally* well-behaved, and is closely related to the critical cases of cut and identity elimination which we may discuss later on. Therefore, when designing or extending a logic, checking harmony is a first step.

From the verificationist standpoint, a connective is *harmonious* if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first condition is called *local soundness* and the second condition is called *local completeness*. The content of the soundness condition is a method to reduce or simplify proofs, and the content of completeness is a method to expand any arbitrary proof into a canonical proof (i.e. one that ends in an introduction rule).

1.1 Conjunction

Local soundness for conjunction is witnessed by the following two reduction rules:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix} \quad \begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}}{A \land B\ \mathit{true}}\land I}{A\ \mathit{true}}\land E_1 \quad \longrightarrow_R \quad \begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix} \quad \begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}}{A \land B\ \mathit{true}}\land I}{B\ \mathit{true}}\land E_2 \quad \longrightarrow_R \quad \begin{matrix}\mathcal{E}\\ B\ \mathit{true}\end{matrix}$$

Local completeness is witnessed by the following expansion rule:

$$\begin{matrix}\mathcal{D}\\ A \land B\ \mathit{true}\end{matrix} \quad \longrightarrow_E \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A \land B\ \mathit{true}\end{matrix}}{A\ \mathit{true}}\land E_1 \quad \dfrac{\begin{matrix}\mathcal{D}\\ A \land B\ \mathit{true}\end{matrix}}{B\ \mathit{true}}\land E_2}{A \land B\ \mathit{true}}\land I$$

When regarded as generating relations on *programs* rather than proofs, the reduction and expansion rules can be recast into another familiar format:

$$\begin{aligned} \mathsf{fst}\,\langle M, N\rangle &\longrightarrow_R M \\ \mathsf{snd}\,\langle M, N\rangle &\longrightarrow_R N \\ M &\longrightarrow_E \langle \mathsf{fst}\,M, \mathsf{snd}\,M\rangle \end{aligned}$$

1.2 Disjunction

Local soundness:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\ \mathit{true}\end{matrix}}{A \lor B\ \mathit{true}}\lor I_1 \quad \begin{matrix}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix} \quad \begin{matrix}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}}{C\ \mathit{true}}\lor E^{u,v} \quad \longrightarrow_R \quad \begin{matrix}\mathcal{D}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ B\ \mathit{true}\end{matrix}}{A \lor B\ \mathit{true}}\lor I_2 \quad \begin{matrix}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{matrix} \quad \begin{matrix}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}}{C\ \mathit{true}}\lor E^{u,v} \quad \longrightarrow_R \quad \begin{matrix}\mathcal{D}\\ \overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{matrix}$$

$$\begin{aligned} \mathsf{case}\ \mathsf{inl}\,M\ \mathsf{of}\ \mathsf{inl}\,u \Rightarrow L \mid \mathsf{inr}\,v \Rightarrow R &\longrightarrow_R [M/u]L \\ \mathsf{case}\ \mathsf{inr}\,M\ \mathsf{of}\ \mathsf{inl}\,u \Rightarrow L \mid \mathsf{inr}\,v \Rightarrow R &\longrightarrow_R [M/v]R \end{aligned}$$

Local completeness:

$$\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \dfrac{\begin{array}{c}\mathcal{D}\\ A\lor B\ \mathit{true}\end{array} \quad \dfrac{\overline{A\ \mathit{true}}^{\,u}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_1 \quad \dfrac{\overline{B\ \mathit{true}}^{\,v}}{A\lor B\ \mathit{true}}\lor\mathsf{I}_2}{A\lor B\ \mathit{true}}\lor\mathsf{E}^{u,v}$$

$$M \longrightarrow_{\mathsf{E}} \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}\,u \Rightarrow \mathsf{inl}\,u \mid \mathsf{inr}\,v \Rightarrow \mathsf{inr}\,v$$

1.3 Implication

Local soundness:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}{A\supset B\ \mathit{true}}\supset\mathsf{I}^u \quad \begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}}{B}\supset\mathsf{E} \longrightarrow_{\mathsf{R}} \begin{array}{c}\dfrac{\mathcal{E}}{A\ \mathit{true}}u\\ \mathcal{D}\\ B\ \mathit{true}\end{array}$$

$$(\mathsf{fn}\ u \Rightarrow M)\ N \longrightarrow_{\mathsf{R}} [N/u]M$$

Local completeness:

$$\begin{array}{c}\mathcal{D}\\ A\supset B\ \mathit{true}\end{array}\longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\A\supset B\ \mathit{true}\end{array}\quad \overline{A\ \mathit{true}}^{\,u}}{B\ \mathit{true}}\supset\mathsf{E}}{A\supset B\ \mathit{true}}\supset\mathsf{I}^u$$

$$M \longrightarrow_{\mathsf{E}} \mathsf{fn}\ u \Rightarrow M\ u$$

1.4 Experiment: Alternative Implication

What if we replaced the ⊃E rule with the following elimination rule:

$$\dfrac{A\supset B\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\supset\mathsf{E}^u$$

The program/proof term assignment is as follows:

$$\dfrac{L : A\supset B \quad M : A \quad \begin{array}{c}\overline{u : B}^{\,u}\\ \vdots\\ N : C\end{array}}{\mathsf{let}\ u = L\,M\ \mathsf{in}\ N : C}\supset\mathsf{E}^u$$

Task 1. Can we show local soundness and completeness for this version of the implication connective?

Solution 1:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,v}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}{A\supset B\ \mathit{true}}\supset\mathsf{I}^v \quad \begin{array}{c}\mathcal{E}\\A\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\supset\mathsf{E}^u \longrightarrow_{\mathsf{R}} \begin{array}{c}\dfrac{\begin{array}{c}\mathcal{E}\\A\ \mathit{true}\end{array}}{A\ \mathit{true}}v\\ \mathcal{D}\\ \dfrac{B\ \mathit{true}}{B\ \mathit{true}}u\\ \mathcal{F}\\ C\ \mathit{true}\end{array}$$

$$\mathsf{let}\ u = (\mathsf{fn}\ v\Rightarrow L)\ M\ \mathsf{in}\ N \longrightarrow_{\mathsf{R}} [[M/v]L/u]N$$

$$\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \quad \dfrac{}{A\ \mathit{true}}u \quad \dfrac{}{B\ \mathit{true}}v}{B\ \mathit{true}}\supset\mathsf{E}^{v}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^{u}$$

$$M \longrightarrow_{\mathsf{E}} \mathsf{fn}\ u \Rightarrow \mathsf{let}\ v = M\ u\ \mathsf{in}\ v$$

Task 2. Last week in recitation we made up introduction and elimination rules to go with a new connective $\curlywedge$. They are listed again below. Show local soundness and completeness of this connective's rules.

$$\dfrac{A\ \mathit{true} \qquad \begin{array}{c}\dfrac{}{B\ \mathit{true}}u\\ \vdots\\ \dfrac{}{\bot\ \mathit{true}}\end{array}}{A \curlywedge B\ \mathit{true}}\curlywedge I_1$$

$$\dfrac{\begin{array}{c}\dfrac{}{A\ \mathit{true}}u\\ \vdots\\ \dfrac{}{\bot\ \mathit{true}}\end{array} \qquad B\ \mathit{true}}{A \curlywedge B\ \mathit{true}}\curlywedge I_2$$

$$\dfrac{A \curlywedge B\ \mathit{true} \qquad \dfrac{\begin{array}{cc}\dfrac{}{A\ \mathit{true}}u & \dfrac{}{\neg B\ \mathit{true}}v\\ \vdots & \vdots\end{array}}{C\ \mathit{true}} \qquad \dfrac{\begin{array}{cc}\dfrac{}{\neg A\ \mathit{true}}w & \dfrac{}{B\ \mathit{true}}x\\ \vdots & \vdots\end{array}}{C\ \mathit{true}}}{C\ \mathit{true}}\curlywedge E$$

Solution 2: Local soundness:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\dfrac{}{B\ \mathit{true}}u\\ \mathcal{E}\\ \bot\ \mathit{true}\end{array}}{A \curlywedge B\ \mathit{true}}\curlywedge\mathsf{I}_1^u \qquad \begin{array}{c}\dfrac{}{A\ \mathit{true}}w \quad \dfrac{}{\neg B\ \mathit{true}}v\\ \mathcal{F}\\ C\ \mathit{true}\end{array} \qquad \begin{array}{c}\dfrac{}{\neg A\ \mathit{true}}y \quad \dfrac{}{B\ \mathit{true}}z\\ \mathcal{G}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\curlywedge\mathsf{E} \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\dfrac{\mathcal{D}}{A\ \mathit{true}}w \quad \dfrac{\mathcal{E}}{\neg B\ \mathit{true}}v\\ \mathcal{F}\\ C\ \mathit{true}\end{array}$$

The proof for the second introduction rule is analogous to the above.

Local completeness:

$$\begin{array}{c}\mathcal{D}\\ A \curlywedge B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \dfrac{\begin{array}{c}\mathcal{D}\\ A \curlywedge B\ \mathit{true}\end{array} \quad \dfrac{\dfrac{}{A\ \mathit{true}}u \quad \dfrac{}{\neg B\ \mathit{true}}v}{A \curlywedge B\ \mathit{true}}\curlywedge\mathsf{I}_1^v \quad \dfrac{\dfrac{}{\neg A\ \mathit{true}}w \quad \dfrac{}{B\ \mathit{true}}x}{A \curlywedge B\ \mathit{true}}\curlywedge\mathsf{I}_2^w}{A \curlywedge B\ \mathit{true}}\curlywedge\mathsf{E}$$

"Proof search" is not a mere matter of practice: it is *praxis*. The dialectic of proof search is to discover ways to pare down the state space of a logic, and then synthesize this into a new logic which is exactly as expressive as the old one. This new restricted logic not only has better search complexity, but also exposes critical semantic content which tends to have been obscured in the original logic.

Perhaps the most famous example of this process is Andreoli's *focalization*; in recent lectures, we have begun to study a simpler instance of this process, namely the decomposition of truth into *verification* and *use*. The passage to verifications constitutes a collation of upward and downward deductions respectively.

2 Verifications and Uses

"Verifications" are proofs that proceed upwards from conclusions to premises; this is also known as *backward inference* or *refinement-style proof*. On the other hand, "uses" are proofs that proceed from premise to conclusion, also known as *forward inference*. The judgment $A \uparrow$ stands for verifications of A, and the judgment $A \downarrow$ stands for uses of A.
The rules for verifications and uses of the conjunction connective are as follows:

$$\frac{A \uparrow \quad B \uparrow}{A \wedge B \uparrow}\ \wedge\mathsf{I} \qquad \frac{A \wedge B \downarrow}{A \downarrow}\ \wedge\mathsf{E}_1 \qquad \frac{A \wedge B \downarrow}{B \downarrow}\ \wedge\mathsf{E}_2$$

On this basis, you may think that verifications correspond to introduction forms and uses correspond to elimination forms. This is not correct, as can be seen from the case of disjunction:

$$\frac{A \uparrow}{A \vee B \uparrow}\ \vee\mathsf{I}_1 \qquad \frac{B \uparrow}{A \vee B \uparrow}\ \vee\mathsf{I}_2 \qquad \frac{A \vee B \downarrow \quad \begin{array}{c}\overline{A \downarrow}^{\,u} \\ \vdots \\ C \uparrow\end{array} \quad \begin{array}{c}\overline{B \downarrow}^{\,v} \\ \vdots \\ C \uparrow\end{array}}{C \uparrow}\ \vee\mathsf{E}^{u,v}$$

Will the elimination rule for implication result have a verification or a use in its conclusion?

$$\frac{\begin{array}{c}\overline{A \downarrow}^{\,u} \\ \vdots \\ B \uparrow\end{array}}{A \supset B \uparrow}\ \supset\mathsf{I}^u \qquad \frac{A \supset B \downarrow \quad A \uparrow}{B \downarrow}\ \supset\mathsf{E}$$

One dimension along which connectives vary is *polarity*: some connectives are positive, and some are negative. We cannot yet make this distinction precise, but some students have already begun to observe it. Later on, we may see that negative connectives have elimination forms as uses, but positive connectives have elimination forms as verifications.

The calculus of verifications and uses has one extra rule which was not visible in the original logic:

$$\frac{A \downarrow}{A \uparrow}\ \updownarrow$$

Would it be reasonable to add the inverse of the above rule, which concludes $A \downarrow$ from $A \uparrow$? What would be the consequences of this?