Lecture Notes on
Linear Inversion

15-317: Constructive Logic
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The previous lecture introduced linear logic, summarized in Figure 1, as a fundamental tool for understanding the change of state in logic. But we did not yet get around to imposing any structure on the proof search question, that is more daunting in linear logic than in intuitionistic propositional logic, because of its need for careful resource management. This lecture studies inversion as a fundamental tool for identifying which linear logic rules can be used without regret in any proof search.

1 Linear Cuts

The fundamental tool for understanding logic that this course considered so far is the admissibility of cuts. This result ended up being used to subsequently justify several other constructions, especially also those that limit proof search without losing completeness. Let's repeat this success for linear logic, if we can. In order to simplify our notation, we only consider the case without unlimited resources Γ here. Hence we write $\Delta \Vdash C$ *true* as short for $\cdot ; \Delta \Vdash C$ *true* and we elide the judgment *res*, because it is the only one in the antecedent Δ.

The fist obstacle is that naive statements of the cut theorem could not possibly transfer to linear logic:

If $\Delta \Vdash A$ *true* and $\Delta, A \Vdash C$ *true* then $\Delta \Vdash C$ *true*?

This cannot be a correct statement, because linear logic carefully tracks all its resources so cannot use Δ twice. Any resources that went into proving that A is true are no longer available to establish C from A. Even if the cut theorem does not hold, a modified version called linear cuts still does.

Judgmental Rules

$$\frac{}{P\ \mathit{res} \Vdash P\ \mathit{true}}\ \mathrm{id} \qquad \frac{A\ \mathit{ures} \in \Gamma \quad \Gamma; \Delta, A\ \mathit{res} \Vdash C\ \mathit{true}}{\Gamma; \Delta \Vdash C\ \mathit{true}}\ \mathrm{copy}$$

Multiplicative Connectives

$$\frac{\Delta_A \Vdash A \quad \Delta_B \Vdash B}{\Delta_A, \Delta_B \Vdash A \otimes B}\ \otimes R \qquad \frac{\Delta, A\ \mathit{res}, B\ \mathit{res} \Vdash C\ \mathit{true}}{\Delta, A \otimes B\ \mathit{res} \Vdash C\ \mathit{true}}\ \otimes L$$

$$\frac{}{\cdot \Vdash \mathbf{1}\ \mathit{true}}\ \mathbf{1}R \qquad \frac{\Delta \Vdash C\ \mathit{true}}{\Delta, \mathbf{1}\ \mathit{res} \Vdash C\ \mathit{true}}\ \mathbf{1}L$$

$$\frac{\Delta, A\ \mathit{res} \Vdash B\ \mathit{true}}{\Delta \Vdash A \multimap B\ \mathit{true}}\ {\multimap}R \qquad \frac{\Delta_A \Vdash A\ \mathit{true} \quad \Delta_B, B\ \mathit{res} \Vdash C\ \mathit{true}}{\Delta_A, \Delta_B, A \multimap B\ \mathit{res} \Vdash C\ \mathit{true}}\ {\multimap}L$$

Additive Connectives

$$\frac{\Delta \Vdash A\ \mathit{true} \quad \Delta \Vdash B\ \mathit{true}}{\Delta \Vdash A \,\&\, B\ \mathit{true}}\ \&R \qquad \frac{\Delta, A\ \mathit{res} \Vdash C\ \mathit{true}}{\Delta, A \,\&\, B\ \mathit{res} \Vdash C\ \mathit{true}}\ \&L_1 \qquad \frac{\Delta, B\ \mathit{res} \Vdash C\ \mathit{true}}{\Delta, A \,\&\, B\ \mathit{res} \Vdash C\ \mathit{true}}\ \&L_2$$

$$\frac{}{\Delta \Vdash \top\ \mathit{true}}\ \top R \qquad \text{no } \top L \text{ rule}$$

$$\frac{\Delta \Vdash A\ \mathit{true}}{\Delta \Vdash A \oplus B\ \mathit{true}}\ \oplus R_1 \qquad \frac{\Delta \Vdash B\ \mathit{true}}{\Delta \Vdash A \oplus B\ \mathit{true}}\ \oplus R_2 \qquad \frac{\Delta, A\ \mathit{res} \Vdash C\ \mathit{true} \quad \Delta, B\ \mathit{res} \Vdash C\ \mathit{true}}{\Delta, A \oplus B\ \mathit{res} \Vdash C\ \mathit{true}}\ \oplus L$$

$$\text{no } \mathbf{0}R \text{ rule} \qquad \frac{}{\Delta, \mathbf{0}\ \mathit{res} \Vdash C\ \mathit{true}}\ \mathbf{0}L$$

Exponential Connective

$$\frac{\Gamma; \cdot \Vdash A\ \mathit{true}}{\Gamma; \cdot \Vdash {!A}\ \mathit{true}}\ {!R} \qquad \frac{(\Gamma, A\ \mathit{ures}); \Delta \Vdash C\ \mathit{true}}{\Gamma; \Delta, {!A}\ \mathit{res} \Vdash C\ \mathit{true}}\ {!L}$$

Figure 1: Intuitionistic Linear Logic

Theorem 1 (Linear cut) *If* $\Delta \Vdash A$ *true and* $\Delta', A\,res \Vdash C$ *true then* $\Delta, \Delta' \Vdash C$ *true.*

Proof: The proof is built in analogy to the cut proof for intuitionistic propositional logic following a similar structure. We only show some cases of the proof. In all uses of the induction hypothesis is it important to appeal to it on strictly smaller instances by a well-founded order. We need to reduce

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\ \mathit{true}\end{array} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Vdash C\ \mathit{true}\end{array} \quad \text{to} \quad \begin{array}{c}\mathcal{F}\\ \Delta, \Delta' \Vdash C\ \mathit{true}\end{array}$$

Case: $\mathcal{D}$ is an initial sequent, $\mathcal{E}$ is arbitrary.

$$\mathcal{D} = \frac{}{A \Vdash A\ \mathit{true}}\ \mathsf{id} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Vdash C\ \mathit{true}\end{array}$$

$\Delta = A$	This case
$\Delta', A \Vdash C\ \mathit{true}$	$\mathcal{E}$
$\Delta, \Delta' \Vdash C\ \mathit{true}$	by above since $\Delta = A$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1 \Vdash A_1\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta_2 \Vdash A_2\ \mathit{true}\end{array}}{\Delta_1, \Delta_2 \Vdash A_1 \otimes A_2\ \mathit{true}}\ \otimes R \quad \text{and} \quad \mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1\\ \Delta', A_1, A_2 \Vdash C\ \mathit{true}\end{array}}{\Delta', A_1 \otimes A_2 \Vdash C\ \mathit{true}}\ \otimes L$$

$\Delta', \Delta_1, A_2 \Vdash C\ \mathit{true}$	By IH on $A_1 \prec A_1 \otimes A_2$, $\mathcal{D}_1 \prec \mathcal{D}$ and $\mathcal{E}_1 \prec \mathcal{E}$
$\Delta', \Delta_1, \Delta_2 \Vdash C\ \mathit{true}$	By IH on $A_2 \prec A_1 \otimes A_2$, $\mathcal{D}_2 \prec \mathcal{D}$, and previous line

At this point in the proof do we again see how important it is that the well-founded order used for induction accepts arbitrarily big deductions as long as the cut formula got smaller.

Case: If $\mathcal{D}$ ended with an $\otimes L$:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Delta, B_1, B_2 \Vdash A\ \mathit{true}\end{array}}{\Delta, B_1 \otimes B_2 \Vdash A\ \mathit{true}}\ \otimes L \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', A \Vdash C\ \mathit{true}\end{array}$$

$\Delta = (\Delta_1, B_1 \otimes B_2)$	This case
$\Delta_1, B_1, B_2, \Delta' \Vdash C\ \mathit{true}$	By IH on A, $\mathcal{D}_1 \prec \mathcal{D}$ and $\mathcal{E}$
$\Delta_1, B_1 \otimes B_2, \Delta' \Vdash C\ \mathit{true}$	By rule $\otimes L$ from above
$\Delta, \Delta' \Vdash C\ \mathit{true}$	Since $\Delta = (\Delta_1, B_1 \otimes B_2)$

Again, we observe how important it is that the induction hypothesis applies if nothing changed except that at least one of the deductions got smaller.

Several other cases are needed to complete the proof of cuts, but this should suffice to give the idea. □

Cuts establish one direction of the correspondence of the resource and truth judgments. Expanding the abbreviated judgments leads to:

If $\Delta \Vdash A\ \textit{true}$ and $\Delta', A\ \textit{res} \Vdash C\ \textit{true}$ then $\Delta, \Delta' \Vdash C\ \textit{true}$.

Hence, if A is proved to be true in linear logic, then it can be used (once!) as a resource. Any C that is provable as a result is also provable directly when assuming the resources Δ that went into proving the truth of A in the first place.

2 Identity

The identity theorem also holds for linear logic. It establishes that every proposition A that is available as a resource can be proved to be true. Taking cuts and identity together allow us to retroactively blur the distinction of the truth and resource judgment somewhat in linear logic, because there is a way to relate them.

Theorem 2 (Identity) *For any proposition A of linear logic: $A \Vdash A\ \textit{true}$.*

Proof: The proof is by induction on the structure of the proof of A.

Case: $A = P$ is an atomic proposition, then the rule for initial sequents suffices directly:

$$\frac{}{P \Vdash P\ \textit{true}}\ \mathrm{id}$$

Case: $A = A_1 \,\&\, A_2$:

$A_1 \Vdash A_1\ \textit{true}$	By IH on $A_1 \prec A_1 \,\&\, A_2$
$A_2 \Vdash A_2\ \textit{true}$	By IH on $A_2 \prec A_1 \,\&\, A_2$
$A_1 \,\&\, A_2 \Vdash A_1\ \textit{true}$	By $\&L_1$ on first line
$A_1 \,\&\, A_2 \Vdash A_2\ \textit{true}$	By $\&L_1$ on second line
$A_1 \,\&\, A_2 \Vdash A_1 \,\&\, A_2\ \textit{true}$	By $\&R$ on last two lines

The other cases of the identity theorem require an inductive proof that will be pursued in the homework. □

3 Inversion

Before we get too carried away with the (exciting) study of metaproperties about linear logic, let's put them to good use in studying more disciplined proof search for linear logic.

Recall from a previous lecture on intuitionistic logic that a proof rule is *invertible* if all premises hold whenever the conclusion does. Invertible proof rules can be applied without regret, because the provability of a sequent is not damaged by trying the rule.

Recall the rule for showing linear implications:

$$\frac{\Delta, A\ \mathit{res} \Vdash B\ \mathit{true}}{\Delta \Vdash A \multimap B\ \mathit{true}} \multimap R$$

Proving that $\multimap R$ is invertible amounts to proving that its inverse is admissible:

$$\frac{\Delta \Vdash A \multimap B\ \mathit{true}}{\Delta, A\ \mathit{res} \Vdash B\ \mathit{true}} \multimap R^{-1}$$

Recall that a proof rule is *admissible* if everything that is provable with the rule is also provable without that rule.

Theorem 3 *Rule $\multimap R$ is invertible, because its inverse is admissible:*

$$\frac{\Delta \Vdash A \multimap B\ \mathit{true}}{\Delta, A\ \mathit{res} \Vdash B\ \mathit{true}} \multimap R^{-1}$$

Proof: A direct proof by induction on the structure of the deduction of $\Delta \Vdash A \multimap B\ \mathit{true}$ would succeed. But this would also be a fair amount of extra work, because all of the proof rules of linear logic need to be considered in the process. And the proof would have to be repeated every time we change the logic. Instead, we will appeal to the cut and identity theorems for help.

$\Delta \Vdash A \multimap B\ \mathit{true}$	By premise of $\multimap R^{-1}$
$A \Vdash A\ \mathit{true}$	By identity
$B \Vdash B\ \mathit{true}$	By identity
$A, A \multimap B \Vdash B\ \mathit{true}$	By $\multimap L$ on previous two lines
$\Delta, A \Vdash B\ \mathit{true}$	By cut with $A \multimap B$ on the first and last line

□

4 Polarity

Now that we have the concept of inversion at our disposal also for linear logic, we can benefit from it in similar ways to intuitionistic logic proof search. Invertible right rules and invertible left rules should be used directly. Rules that require a choice that might turn out to be wrong will be postponed instead. In fact, since, unlike intuitionistic rules, linear rules consume the original formula, we will not have to worry too much about going in never-ending circles during proof search.

Now even if that is the case with the individual rules, we may still end up not being able to prove properties because resources were partitioned inadequately. The right rule for the exponential connective is invertible.

Theorem 4 *Rule $!R$ is invertible:*

$$\frac{\Gamma; \cdot \Vdash A \; \mathit{true}}{\Gamma; \cdot \Vdash\, !A \; \mathit{true}} \; !R$$

Proof: Prove that the inverse rule $!R^{-1}$ of $!R$ is admissible:

$\Gamma; \cdot \Vdash\, !A \; \mathit{true}$	By premise of $!R^{-1}$
$\Gamma, A; A \Vdash A \; \mathit{true}$	By identity
$\Gamma, A; \cdot \Vdash A \; \mathit{true}$	By copy rule on previous line
$\Gamma; !A \Vdash A \; \mathit{true}$	By $!R$ on previous line
$\Gamma; \cdot \Vdash A \; \mathit{true}$	By cut with $!A$ on the first and last line

□

But the rule also requires the resource context Δ to be empty. The rule is perfectly invertible *if* that is the case, however and just not applicable otherwise. Unfortunately, a sequent $\Delta \Vdash\, !A \; \mathit{true}$ with a nonempty linear resource context Δ is not necessarily nonprovable either. For example, a different order of using proof rules would work when Δ contains resources that can still be consumed and end up yielding an empty context.

As previously, we now call connectives *negative* if their right rule is invertible, so they can be used immediately. We call them *positive* if their left rule is invertible, so they can be used immediately.

Negative (R-invertible)	A^-	$::=$	$A \multimap B \mid A \,\&\, B \mid \top$
Positive (L-invertible)	A^+	$::=$	$A \otimes B \mid A \oplus B \mid \mathbf{1} \mid \mathbf{0} \mid\, !A$
Formulas	A	$::=$	$A^- \mid A^+$

At this point, we should no longer be shocked to hear that atomic propositions P have no natural polarity but could be chosen either way.