RECITATION 5: SEQUENT CALCULUS AND KEYMAERA I INTRODUCTION

Yesterday, we presented the sequent calculus as a formalism that will be useful in proof search. We saw that there is a tight correspondence between its rules, and the rules for giving verifications for natural deduction. Today, we will review the rules for the sequent calculus and work through a few example proofs.

1. THE RULES

Recall that left rules correspond to “upside down elimination rules” and that right rules correspond to introduction rules.

\[
\begin{align*}
\Gamma, A \wedge B, A &\Rightarrow C & &\wedge L_1 \\
\Gamma, A \wedge B &\Rightarrow C & &\wedge L_2 \\
\Gamma &\Rightarrow A & &\wedge R \\
\Gamma, A \wedge B &\Rightarrow C & &\wedge L \end{align*}
\]

\[
\begin{align*}
\Gamma, A \vee B, A &\Rightarrow C & &\vee L \\
\Gamma, A \vee B, B &\Rightarrow C & &\vee L_1 \\
\Gamma &\Rightarrow A & &\vee R_1 \\
\Gamma &\Rightarrow A \vee B & &\vee R_2 \\
\Gamma, A \supset B &\Rightarrow A & &\supset L \\
\Gamma, A \supset B, B &\Rightarrow C & &\supset R \\
\Gamma, A &\Rightarrow B & &\supset R \\
\Gamma, A &\Rightarrow A \quad \text{id} \\
\Gamma &\Rightarrow \top & &\top L \\
\Gamma, \bot &\Rightarrow C & &\bot L \\
\Gamma &\Rightarrow \bot & &\bot R \end{align*}
\]

2. KEYMAERA I INTRODUCTION

[Demo of how to set up and use KeYmaera I]

3. SOME EXAMPLE PROOFS

We will spend the remainder of the recitation working through some example proofs, both by hand and in KeYmaera I.\footnote{Most example problems are taken from 15-317 (Fall 2015) Recitation 5, whose notes were prepared by Evan Cavallo, Oliver Daids, and Giselle Reis. Responsibility for any errors herein lies with the present author.}

**Exercise 1.** \( \cdot \Rightarrow A \supset A \)

**Proof.**

\[
\begin{align*}
A &\Rightarrow A \quad \text{id} \\
\cdot &\Rightarrow A \supset A \quad \supset R \end{align*}
\]

Date: 19 February 2020.
Exercise 2. \( \cdot \Rightarrow A \land B \supset B \land A \)

Proof.

\[
\begin{align*}
\forall A \land B, B \Rightarrow B & \quad \text{id} \\
\forall A \land B \Rightarrow B & \quad \text{L1} \\
\forall A \land B & \quad \text{L2} \\
\therefore A \land B & \quad \text{R}
\end{align*}
\]

Exercise 3. \( \cdot \Rightarrow (A \supset (B \land C)) \supset (A \supset B) \)

Proof.

\[
\begin{align*}
\forall (A \supset (B \land C)), A \Rightarrow A & \quad \text{id} \\
\forall (A \supset (B \land C)), A \land B \land C, B \Rightarrow B & \quad \text{id} \\
\forall (A \supset (B \land C)), A \land B & \quad \text{L} \\
\forall (A \supset (B \land C)) \Rightarrow (A \supset B) & \quad \text{R} \\
\therefore (A \supset (B \land C)) \supset (A \supset B) & \quad \text{R}
\end{align*}
\]

Exercise 4. \( \cdot \Rightarrow (A \supset B \supset C) \supset B \supset A \supset C \)

Proof.

\[
\begin{align*}
\forall A \supset B \supset C, B, A \supset B \supset C \Rightarrow B & \quad \text{id} \\
\forall A \supset B \supset C, B, A \supset B \supset C & \quad \text{id} \\
\forall A \supset B \supset C, B & \quad \text{L} \\
\forall A \supset B \supset C, B \Rightarrow A \supset C & \quad \text{R} \\
\forall A \supset B \supset C \Rightarrow B \supset A \supset C & \quad \text{R} \\
\therefore (A \supset B \supset C) \supset B \supset A \supset C & \quad \text{R}
\end{align*}
\]

Exercise 5. \( \cdot \Rightarrow (A \supset B) \supset ((A \supset C) \supset (B \land C)) \)

Proof.

\[
\begin{align*}
\forall (A \supset B), (A \supset C), A \Rightarrow A & \quad \text{id} \\
\forall (A \supset B), (A \supset C), (A \supset B), (A \supset C) \Rightarrow B & \quad \text{id} \\
\forall (A \supset B), (A \supset C) & \quad \text{L} \\
\forall (A \supset B), (A \supset C) & \quad \text{R} \\
\forall (A \supset B), (A \supset C) \Rightarrow B \land C & \quad \text{L1} \\
\forall (A \supset B), (A \supset C) & \quad \text{R} \\
\forall (A \supset B) & \quad \text{L2} \\
\forall (A \supset B) \Rightarrow ((A \supset C) \supset (B \land C)) & \quad \text{R} \\
\therefore (A \supset B) \supset ((A \supset C) \supset (B \land C)) & \quad \text{R}
\end{align*}
\]