Constructive Logic (15-317), Spring 2020
Midterm Review Problems

Instructor: André Platzer
TAs: Avery Cowan, Cameron Wong, Carter Williams, Klaas Pruiksma

February 11, 2020

1 Terms and Conditions

Task 1. Give the natural deduction tree corresponding to the proof term
$\texttt{fn}(f : A \supset A) \Rightarrow \texttt{fn}(x : A) \Rightarrow (\texttt{fn}(y : A) \Rightarrow x)(f\ x)$

In each of the following tasks, let P be the proposition $\neg\neg(((A \supset B) \supset A) \supset A)$.

Task 2. Give a derivation of $P\ \mathit{true}$ that cannot be directly converted to a verification (that is, there is no valid assignment of arrows and insertions of ↓↑ satisfying the usual rules of verifications and uses). Also, give its corresponding proof term.

Task 3. Give a verification of P (derive $P \uparrow$) and its corresponding proof term.

Task 4. What is the relationship between the proof terms you gave in tasks 2 and 3?

2 Objective Connectives

Let the $*$ connective (not to be confused with the $*$ connective from the homework!) be defined by the following rule:

$$\dfrac{A * B\ \mathit{true} \qquad \begin{array}{c}\overline{A\ \mathit{true}}\ u \\ \vdots \\ B \supset C\ \mathit{true}\end{array} \qquad \begin{array}{c}\overline{C\ \mathit{true}}\ v \\ \vdots \\ D\ \mathit{true}\end{array}}{C \wedge D\ \mathit{true}}\ *E^{u,v}$$

Task 5. Come up with an appropriate introduction rule, and prove that it is harmonious with $*E^{u,v}$. You should produce exactly one rule, which should not use connectives other than $*$.

Task 6. The supposed elimination rule, $*E^{u,v}$ is not a "proper" elimination rule, as it uses other connectives, namely $\supset$ and $\wedge$. Give exactly one replacement rule, harmonious with the introduction rule given in the previous task (you don't need to prove this), that does not use other connectives.

Task 7. It turns out that $*$ is equivalent to another connective we have already seen. Give that connective, then prove that they are equivalent. That is, if you think that $A * B$ is equivalent to $A \vee B$, you should show that $(A * B) \supset (A \vee B)\ \mathit{true}$ and $(A \vee B) \supset (A * B)\ \mathit{true}$.

3 Arithmetic

For your convenience, here are the rules of Heyting Arithmetic:

$$\frac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \frac{x : \mathsf{nat}}{\mathsf{s}\ x : \mathsf{nat}}\ \mathsf{nat}I_S$$

$$\frac{x : \mathsf{nat} \quad C(0)\ \mathit{true} \quad \begin{array}{c} \overline{y : \mathsf{nat}} \quad \overline{C(y)\ \mathit{true}} \\ \vdots \\ C(\mathsf{s}\ y)\ \mathit{true} \end{array}}{C(x)\ \mathit{true}}\ \mathsf{nat}E^{y,u}$$

$$\frac{}{0 = 0\ \mathit{true}} = I_{00} \qquad \frac{x = y\ \mathit{true}}{\mathsf{s}\ x = \mathsf{s}\ y\ \mathit{true}} = I_{SS}$$

$$\frac{0 = \mathsf{s}\ x\ \mathit{true}}{C\ \mathit{true}} = E_{0S} \qquad \frac{\mathsf{s}\ x = 0\ \mathit{true}}{C\ \mathit{true}} = E_{S0} \qquad \frac{\mathsf{s}\ x = \mathsf{s}\ y\ \mathit{true}}{x = y\ \mathit{true}} = E_{SS}$$

$$\frac{}{R(0; t_0; x, r.t_S) \Rightarrow_R t_0} \Rightarrow_R I_0 \qquad \frac{}{R(\mathsf{s}\ n; t_0; x, r.t_S) \Rightarrow_R [R(n; t_0; x, r.t_S)/r][n/x]t_S} \Rightarrow_R I_S$$

$$\frac{A(x)\ \mathit{true} \quad x \Rightarrow_R y}{A(y)\ \mathit{true}} \Rightarrow_R E_1 \qquad \frac{A(y)\ \mathit{true} \quad x \Rightarrow_R y}{A(x)\ \mathit{true}} \Rightarrow_R E_2$$

Task 8. Prove $\forall(n : \mathsf{nat}).(n = 0) \vee \exists(m : \mathsf{nat}).(\mathsf{s}\ m = n \wedge \forall(p : \mathsf{nat}).(\mathsf{s}\ p = n \supset p = m))\ \mathit{true}$. That is, show that every natural number is either zero or has a unique predecessor.

Next, let us extend these with two new predicates, even and odd, defined as follows:

$$\frac{}{\mathit{even}(0)\ \mathit{true}}\ \mathit{even}_0 \qquad \frac{n : \mathsf{nat} \quad \mathit{even}(n)\ \mathit{true}}{\mathit{odd}(\mathsf{s}\ n)\ \mathit{true}}\ \mathit{odd} \qquad \frac{n : \mathsf{nat} \quad \mathit{odd}(n)\ \mathit{true}}{\mathit{even}(\mathsf{s})\ \mathit{true}}\ \mathit{even}_S$$

Task 9. Let $\mathit{double}(n)$ be defined as $R(n; 0; x, r.\mathsf{s}(\mathsf{s}\ r))$. Prove $\forall(n : \mathsf{nat}).\mathit{even}(\mathit{double}(n))\ \mathit{true}$.