Recitation 2: Harmony

Course Staff

Proof-theoretic harmony is a necessary, but not sufficient, condition for the well-behavedness of a logic; harmony ensures that the connectives are *locally* well-behaved, and is closely related to the critical cases of cut and identity elimination which we may discuss later on. Therefore, when designing or extending a logic, checking harmony is a first step.

From the verificationist standpoint, a connective is *harmonious* if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first condition is called *local soundness* and the second condition is called *local completeness*. The content of the soundness condition is a method to reduce or simplify proofs, and the content of completeness is a method to expand any arbitrary proof into a canonical proof (i.e. one that ends in an introduction rule).

1 Conjunction

Local soundness for conjunction is witnessed by the following two reduction rules:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}}{A\ \mathit{true}}\wedge\mathsf{E}_1 \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}}{B\ \mathit{true}}\wedge\mathsf{E}_2 \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}$$

Local completeness is witnessed by the following expansion rule:

$$\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array} \quad \longrightarrow_{\mathsf{E}} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{A\ \mathit{true}}\wedge\mathsf{E}_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{B\ \mathit{true}}\wedge\mathsf{E}_2}{A \wedge B\ \mathit{true}}\wedge\mathsf{I}$$

When regarded as generating relations on *programs* rather than proofs, the reduction and expansion rules can be recast into another familiar format:

$$\begin{aligned} \pi_1(\langle M, N\rangle) &\longrightarrow_{\mathsf{R}} M \\ \pi_2(\langle M, N\rangle) &\longrightarrow_{\mathsf{R}} N \\ M &\longrightarrow_{\mathsf{E}} \langle \pi_1(M), \pi_2(M)\rangle \end{aligned}$$

2 Disjunction

Instructions: present local soundness for proofs, and ask the students to come up with the version for programs. Next, elicit from the students both local completeness for programs and for proofs.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_1 \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\lor\mathsf{E}^{u,v} \longrightarrow_{\mathsf{R}} \begin{array}{c}\mathcal{D}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ B\ \mathit{true}\end{array}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_2 \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\lor\mathsf{E}^{u,v} \longrightarrow_{\mathsf{R}} \begin{array}{c}\mathcal{D}\\ \overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}$$

$$\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow L \mid \mathsf{inr}(v) \Rightarrow R \longrightarrow_{\mathsf{R}} [M\ /\ u]L$$

$$\mathsf{case}\ \mathsf{inr}(M)\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow L \mid \mathsf{inr}(v) \Rightarrow R \longrightarrow_{\mathsf{R}} [M\ /\ v]R$$

$$\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \dfrac{\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array} \quad \dfrac{A\ \mathit{true}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_1 \quad \dfrac{B\ \mathit{true}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_2}{A \lor B\ \mathit{true}}\lor\mathsf{E}^{u,v}$$

$$M \longrightarrow_{\mathsf{E}} \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow \mathsf{inl}(u) \mid \mathsf{inr}(v) \Rightarrow \mathsf{inr}(v)$$

3 Implication

Elicit both local soundness and local completeness from students in both proof and program notation.

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^u \quad \begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}}{B}\supset\mathsf{E} \longrightarrow_{\mathsf{R}} \begin{array}{c}\mathcal{E}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}$$

$$(\lambda u.\ M)(N) \longrightarrow_{\mathsf{R}} [N\ /\ u]M$$

$$\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array} \quad \overline{A\ \mathit{true}}^{\,u}}{B\ \mathit{true}}\supset\mathsf{E}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^u$$

$$M \longrightarrow_{\mathsf{E}} \lambda u.\ M(u)$$

4 Experiment: Alternative Implication

What if we replaced the ⊃E rule with the following elimination rule:

$$\dfrac{A \supset B\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u} \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ {\supset}\mathsf{E}^u$$

The program/proof term assignment is as follows:

$$\dfrac{L : A \supset B \quad M : A \quad \begin{array}{c}\overline{u : B}^{\,u} \\ \vdots \\ N : C\end{array}}{\mathsf{let}\ u = L(M)\ \mathsf{in}\ N : C}\ {\supset}\mathsf{E}^u$$

Can we show local soundness and completeness for this version of the implication connective?

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,v} \\ \mathcal{D} \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ {\supset}\mathsf{I}^v \quad \begin{array}{c}\mathcal{E} \\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u} \\ \mathcal{F} \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ {\supset}\mathsf{E}^u \ \longrightarrow_{\mathsf{R}}\ \begin{array}{c}\mathcal{E} \\ \overline{A\ \mathit{true}}^{\,v} \\ \mathcal{D} \\ \overline{B\ \mathit{true}}^{\,u} \\ \mathcal{F} \\ C\ \mathit{true}\end{array}$$

$$\mathsf{let}\ u = \lambda v.\, L(M)\ \mathsf{in}\ N \ \longrightarrow_{\mathsf{R}}\ [[M\ /\ v]L\ /\ u]N$$

$$\begin{array}{c}\mathcal{D} \\ A \supset B\ \mathit{true}\end{array} \ \longrightarrow_{\mathsf{E}}\ \dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ A \supset B\ \mathit{true}\end{array} \quad \overline{A\ \mathit{true}}^{\,u} \quad \overline{B\ \mathit{true}}^{\,v}}{B\ \mathit{true}}\ {\supset}\mathsf{E}^v}{A \supset B\ \mathit{true}}\ {\supset}\mathsf{I}^u$$

$$M \ \longrightarrow_{\mathsf{E}}\ \lambda u.\ \mathsf{let}\ v = M(u)\ \mathsf{in}\ v$$