Midterm I Exam

15-317/657 Constructive Logic
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Name: ____________________

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Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 6 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
New Connections	45	
Harmonic Series	20	
Proof Terms	55	
VerifyThis Quantifier	30	
Total:	150	

1 New Connections (45 points)

Consider the new connective $\Box(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$\dfrac{\begin{array}{cc} \dfrac{}{A\ \mathit{true}}u & \dfrac{}{B\ \mathit{true}}w \\ \vdots & \vdots \\ B\ \mathit{true} & C\ \mathit{true} \end{array}}{\Box(A, B, C)\ \mathit{true}}\ \Box I^{u,w}$$

$\boxed{10}$ **Task 1** Give a set of elimination rules that harmoniously fit to $\Box I$:

$\boxed{10}$ **Task 2** Prove local soundness for the $\Box$ connective.

$\boxed{10}$ **Task 3** Prove local completeness for the $\Box$ connective.

$\boxed{5}$ **Task 4** Propose a proof term assignment for $\Box I^{u,w}$ **and** all other rules of the $\Box$ connective.

$$\dfrac{\begin{array}{cc}\overline{\qquad : A\qquad}^{\,u} & \overline{\qquad : B\qquad}^{\,w} \\ \vdots & \vdots \\ : B & : C\end{array}}{: \Box(A,B,C)}\ \Box I^{u,w}$$

$\boxed{10}$ **Task 5** Provide all local reduction rules for the proof terms of the $\Box$ connective.

2 Harmonic Series (20 points)

This question considers introduction and elimination rules. Mark connectives as:

- ⓗ for harmonious connectives and provide local reductions on proofs. You do not need to give local expansions (but convince yourself it is locally sound and locally complete).
- ⓤ for unharmonious connectives and briefly explain one case that fails and why.

10 **Task 1**

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \mathrel{-\!\!<} B\ \mathit{true}} \mathrel{-\!\!<}I \qquad \frac{A \mathrel{-\!\!<} B\ \mathit{true} \quad \begin{array}{c} \overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ C\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}^{\,w} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}} \mathrel{-\!\!<}E^{u,w}$$

10 **Task 2**

$$\frac{\begin{array}{c} \overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ B\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}^{\,w} \\ \vdots \\ A\ \mathit{true} \end{array}}{A \mathrel{>\!\!<} B\ \mathit{true}} >\!\!<I^{u,w} \qquad \frac{A \mathrel{>\!\!<} B\ \mathit{true} \quad B\ \mathit{true}}{A\ \mathit{true}} >\!\!<E_1 \qquad \frac{A \mathrel{>\!\!<} B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}} >\!\!<E_2$$

3 Proof Terms (55 points)

This question studies the proof terms of natural deduction. Recall that a proof term is called *normal*/irreducible if it cannot be reduced by any local reduction of proof terms.

10 **Task 1** Give a **normal** proof term justifying that $\big((A \supset B) \wedge (A \supset C)\big) \supset \big(A \supset (B \wedge C)\big)$ is true.

10 **Task 2** What proposition is the following proof term justifying?
$\mathsf{fn}\ u \Rightarrow \mathbf{fst}\big\langle \mathsf{fn}\ v \Rightarrow \mathbf{snd}(u\,v)\ ,\ \mathbf{case}\ \mathbf{inl}\ u\ \mathbf{of}\ \mathbf{inl}\ w \Rightarrow (\mathsf{fn}\ y \Rightarrow \mathbf{fst}(u\,y))\mid \mathbf{inr}\ z \Rightarrow \mathbf{abort}\ z\big\rangle$

10 **Task 3** Give a **normal** proof term justifying the proposition from Task 2 **or** explain why that proof term already is normal.

5 **Task 4** Give a **normal** proof term that justifies that $A \supset (A \vee B)$ is true.

10 **Task 5** Give a proof term that is **not** normal but justifies that $A \supset (A \vee B)$ is true.

10 **Task 6** Recall primitive recursion terms $R(n, t_0, x.r.t_s)$ for natural number n that are generated by successor s from $\mathsf{0}$. What **normal** proof term does the following proof term reduce to?
$R(\mathsf{s}\,\mathsf{s}\,\mathsf{0}, a, x.r.f(r, x, r))$

4 VerifyThis Quantifier (30 points)

Fill in missing **propositions** (if any), verification/use **judgments** (↑ and ↓) and **inference rule names** that make the following figures correct verifications **or** explain why that is **impossible** to do correctly. As in lecture, you do not need to decorate typing judgments $a : \tau$ with ↑ or ↓.

$\boxed{15}$ **Task 1**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x{:}\tau.\ (A(x) \wedge B(x))}^{\,u} \qquad \overline{a : \tau}}{\qquad\qquad}}{\qquad\qquad}}{A(a)}}{\forall x{:}\tau.\ A(x)}}{(\forall x{:}\tau.\ (A(x) \wedge B(x))) \supset (\forall x{:}\tau.\ A(x))}
$$

$\boxed{15}$ **Task 2**

$$
\dfrac{\dfrac{\overline{\exists x{:}\tau.\ C(x)}^{\,u} \qquad \overline{a : \tau} \quad \dfrac{\overline{C(a)}^{\,w}}{C(a)}}{C(a)}}{(\exists x{:}\tau.\ C(x)) \supset C(a)}
$$