Midterm I Exam

15-317/657 Constructive Logic
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Name: ____________________________________________________________

Andrew ID: _______________________________________________________

Instructions

• This exam is closed-book with one sheet of notes permitted.
• You have 80 minutes to complete the exam.
• There are 4 problems on 6 pages.
• Read each problem carefully before attempting to solve it.
• Do not spend too much time on any one problem.
• Consider if you might want to skip a problem on a first pass and return to it later.

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Max Score: 150
1 New Connections (45 points)

Consider the new connective $\square(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$
\frac{\begin{array}{c}
A \text{ true} \ \ \ B \text{ true} \\
\vdots \\
B \text{ true} \ \ C \text{ true}
\end{array}}{\square(A, B, C) \text{ true}} \quad \square^u,w
$$

**Task 1** Give a set of elimination rules that harmoniously fit to $\square I$:

**Task 2** Prove local soundness for the $\square$ connective.
Task 3 Prove local completeness for the □ connective.

Task 4 Propose a proof term assignment for □I^{u,w} and all other rules of the □ connective.

\[
\begin{array}{c}
\vdash A^u \\
\vdash B^w \\
\vdash B \\
\vdash C \\
\vdash \Box (A, B, C) \\
\end{array}
\]

Task 5 Provide all local reduction rules for the proof terms of the □ connective.
2 Harmonic Series (20 points)

This question considers introduction and elimination rules. Mark connectives as:

- for harmonious connectives and provide local reductions on proofs. You do not need to give local expansions (but convince yourself it is locally sound and locally complete).
- for unharmonious connectives and briefly explain one case that fails and why.

10 Task 1

\[
\[
\[
\begin{array}{c}
A \text{ true} \\
B \text{ true}
\end{array} 
\quad <I
\quad A \rightarrow B \text{ true} \\
C \text{ true}
\quad <E^{u,w}
\]

10 Task 2

\[
\[
\[
\begin{array}{c}
A \text{ true} \\
B \text{ true}
\end{array} 
\quad <I^{u,w}
\quad A \nsucceq B \text{ true} \\
A \text{ true}
\quad \times E_1
\quad A \nsucceq B \text{ true} \\
B \text{ true}
\quad \times E_2
\]

\]
3 Proof Terms (55 points)
This question studies the proof terms of natural deduction. Recall that a proof term is called normal/irreducible if it cannot be reduced by any local reduction of proof terms.

10 Task 1 Give a normal proof term justifying that \((A \supset B) \land (A \supset C) \supset (A \supset (B \land C))\) is true.

10 Task 2 What proposition is the following proof term justifying?
\[
\text{fn } u \Rightarrow \text{fst}(\text{fn } v \Rightarrow \text{snd}(u v), \text{case inl } u \text{ of inl } w \Rightarrow (\text{fn } y \Rightarrow \text{fst}(u y)) | \text{inr } z \Rightarrow \text{abort } z)
\]

10 Task 3 Give a normal proof term justifying the proposition from Task 2 or explain why that proof term already is normal.

5 Task 4 Give a normal proof term that justifies that \(A \supset (A \lor B)\) is true.

10 Task 5 Give a proof term that is not normal but justifies that \(A \supset (A \lor B)\) is true.

10 Task 6 Recall primitive recursion terms \(R(n, t_0, x.r.t_s)\) for natural number \(n\) that are generated by successor \(s\) from 0. What normal proof term does the following proof term reduce to?
\[
R(s \ s \ 0, a, x.r.f(r, x, r))
\]
4 Verify This Quantifier (30 points)

Fill in missing propositions (if any), verification/use judgments (↑ and ↓) and inference rule names that make the following figures correct verifications or explain why that is impossible to do correctly. As in lecture, you do not need to decorate typing judgments $a : \tau$ with ↑ or ↓.

**Task 1**

$$
\frac{\forall x : \tau. \ (A(x) \land B(x))}{A(a)}
\quad \frac{\forall x : \tau. \ A(x)}{(\forall x : \tau. \ (A(x) \land B(x))) \supset (\forall x : \tau. \ A(x))}
\quad \frac{a : \tau}{\forall x : \tau. \ A(x)}
\quad \frac{u}{a : \tau}
$$

**Task 2**

$$
\frac{\exists x : \tau. \ C(x)}{C(a)}
\quad \frac{C(a)}{\exists x : \tau. \ C(x) \supset C(a)}
$$