Midterm I Exam

15-317/657 Constructive Logic
André Platzer

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Name: André Platzer

Andrew ID: aplatzer

Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 6 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
New Connections	45	
Harmonic Series	20	
Proof Terms	55	
VerifyThis Quantifier	30	
Total:	150	

Please keep in mind that this is a sample solution, *not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.*

1 New Connections (45 points)

Consider the new connective $\square(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,w}\\ \vdots & \vdots \\ B\ \mathit{true} & C\ \mathit{true}\end{array}}{\square(A,B,C)\ \mathit{true}}\ \square I^{u,w}$$

10 **Task 1** Give a set of elimination rules that harmoniously fit to $\square I$:

Solution:

$$\dfrac{\square(A,B,C)\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\ \square E_1 \qquad \dfrac{\square(A,B,C)\ \mathit{true} \quad B\ \mathit{true}}{C\ \mathit{true}}\ \square E_2$$

10 **Task 2** Prove local soundness for the $\square$ connective.

Solution:

$$\dfrac{\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,w}\\ \mathcal{D} & \mathcal{E} \\ B\ \mathit{true} & C\ \mathit{true}\end{array}}{\square(A,B,C)\ \mathit{true}}\ \square I^{u,w} \quad \begin{array}{c}\mathcal{F}\\ A\ \mathit{true}\end{array}}{B\ \mathit{true}}\ \square E_1 \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{F}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,w}\\ \mathcal{D} & \mathcal{E} \\ B\ \mathit{true} & C\ \mathit{true}\end{array}}{\square(A,B,C)\ \mathit{true}}\ \square I^{u,w} \quad \begin{array}{c}\mathcal{F}\\ B\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \square E_2 \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{F}\\ \overline{B\ \mathit{true}}^{\,w}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}$$

10 **Task 3** Prove local completeness for the $\Box$ connective.

Solution:

$$\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \Rightarrow_E \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \quad \dfrac{}{A\ \mathit{true}}u}{B\ \mathit{true}}\Box E_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \quad \dfrac{}{B\ \mathit{true}}w}{C\ \mathit{true}}\Box E_2}{\Box(A,B,C)\ \mathit{true}}\Box I^{u,w}$$

5 **Task 4** Propose a proof term assignment for $\Box I^{u,w}$ **and** all other rules of the $\Box$ connective.

$$\dfrac{\begin{array}{c}\overline{u : A}\,u\\ \vdots\\ M : B\end{array} \quad \begin{array}{c}\overline{w : B}\,w\\ \vdots\\ N : C\end{array}}{trans(u.M, w.N) : \Box(A,B,C)}\Box I^{u,w}$$

Solution:

$$\dfrac{M : \Box(A,B,C) \quad N : A}{tfs(M,N) : B}\Box E_1 \qquad \dfrac{M : \Box(A,B,C) \quad N : B}{tsn(M,N) : C}\Box E_2$$

10 **Task 5** Provide all local reduction rules for the proof terms of the $\Box$ connective.

Solution:

$$tfs(trans(u.M, w.N), O) \Rightarrow_R [O/u]M$$
$$tsn(trans(u.M, w.N), O) \Rightarrow_R [O/w]N$$

2 Harmonic Series (20 points)

This question considers introduction and elimination rules. Mark connectives as:

ⓗ for harmonious connectives and provide local reductions on proofs. You do not need to give local expansions (but convince yourself it is locally sound and locally complete).

ⓤ for unharmonious connectives and briefly explain one case that fails and why.

10 **Task 1**

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \mathrel{-\!\!<} B\ \mathit{true}} \mathrel{-\!\!<}I \qquad \frac{A \mathrel{-\!\!<} B\ \mathit{true} \quad \begin{array}{c} \overline{A\ \mathit{true}}\ u \\ \vdots \\ C\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}\ w \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}} \mathrel{-\!\!<}E^{u,w}$$

Solution: ⓤ incomplete since only facts C can be derived by elimination that can be deduced individually from A as well as from B but not from both together as the introduction rule $\mathrel{-\!\!<}I$ requires.

10 **Task 2**

$$\frac{\begin{array}{c} \overline{A\ \mathit{true}}\ u \\ \vdots \\ B\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}\ w \\ \vdots \\ A\ \mathit{true} \end{array}}{A \mathrel{>\!\!<} B\ \mathit{true}} \mathrel{>\!\!<}I^{u,w} \qquad \frac{A \mathrel{>\!\!<} B\ \mathit{true} \quad B\ \mathit{true}}{A\ \mathit{true}} \mathrel{>\!\!<}E_1 \qquad \frac{A \mathrel{>\!\!<} B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}} \mathrel{>\!\!<}E_2$$

Solution: ⓗ

$$\frac{\dfrac{\begin{array}{c} \overline{A\ \mathit{true}}\ u \\ \mathcal{D} \\ B\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}\ w \\ \mathcal{E} \\ A\ \mathit{true} \end{array}}{A \mathrel{>\!\!<} B\ \mathit{true}} \mathrel{>\!\!<}I^{u,w} \quad \begin{array}{c} \mathcal{F} \\ B\ \mathit{true} \end{array}}{A\ \mathit{true}} \mathrel{>\!\!<}E_1 \quad \Longrightarrow_R \quad \begin{array}{c} \mathcal{F} \\ \overline{B\ \mathit{true}}\ w \\ \mathcal{E} \\ A\ \mathit{true} \end{array}$$

$$\frac{\dfrac{\begin{array}{c} \overline{A\ \mathit{true}}\ u \\ \mathcal{D} \\ B\ \mathit{true} \end{array} \quad \begin{array}{c} \overline{B\ \mathit{true}}\ w \\ \mathcal{E} \\ A\ \mathit{true} \end{array}}{A \mathrel{>\!\!<} B\ \mathit{true}} \mathrel{>\!\!<}I^{u,w} \quad \begin{array}{c} \mathcal{F} \\ A\ \mathit{true} \end{array}}{B\ \mathit{true}} \mathrel{>\!\!<}E_2 \quad \Longrightarrow_R \quad \begin{array}{c} \mathcal{F} \\ \overline{A\ \mathit{true}}\ u \\ \mathcal{D} \\ B\ \mathit{true} \end{array}$$

3 Proof Terms (55 points)

This question studies the proof terms of natural deduction. Recall that a proof term is called *normal*/irreducible if it cannot be reduced by any local reduction of proof terms.

10 **Task 1** Give a **normal** proof term justifying that $\big((A \supset B) \wedge (A \supset C)\big) \supset \big(A \supset (B \wedge C)\big)$ is true.

> **Solution:** $\mathsf{fn}\, u \Rightarrow \mathsf{fn}\, v \Rightarrow \langle (\mathbf{fst}\, u)v, (\mathbf{snd}\, u)v \rangle$

10 **Task 2** What proposition is the following proof term justifying?
$\mathsf{fn}\ u \Rightarrow \mathbf{fst}\big\langle \mathsf{fn}\ v \Rightarrow \mathbf{snd}(u\,v)\ ,\ \mathbf{case}\,\mathbf{inl}\,u\ \mathbf{of}\ \mathbf{inl}\,w \Rightarrow (\mathsf{fn}\ y \Rightarrow \mathbf{fst}(u\,y)) \mid \mathbf{inr}\,z \Rightarrow \mathbf{abort}\,z \big\rangle$

> **Solution:** $(A \supset E \wedge B) \supset (A \supset B)$ *true*

10 **Task 3** Give a **normal** proof term justifying the proposition from Task 2 **or** explain why that proof term already is normal.

> **Solution:** $\mathsf{fn}\, u \Rightarrow \mathsf{fn}\, v \Rightarrow \mathbf{snd}(u\, v)$

5 **Task 4** Give a **normal** proof term that justifies that $A \supset (A \vee B)$ is true.

> **Solution:** $\mathsf{fn}\, u \Rightarrow \mathbf{inl}_B\, u$

10 **Task 5** Give a proof term that is **not** normal but justifies that $A \supset (A \vee B)$ is true.

> **Solution:** $\mathsf{fn}\, u \Rightarrow \mathbf{fst}\langle \mathbf{inl}_B\, u, \mathbf{inl}_B\, u \rangle$

10 **Task 6** Recall primitive recursion terms $R(n, t_0, x.r.t_s)$ for natural number n that are generated by successor s from 0. What **normal** proof term does the following proof term reduce to?
$R(\mathsf{s}\,\mathsf{s}\,0, a, x.r.f(r, x, r))$

> **Solution:** $f\big(f(a, 0, a), \mathsf{s}\,0, f(a, 0, a)\big)$

4 VerifyThis Quantifier (30 points)

Fill in missing **propositions** (if any), verification/use **judgments** ($\uparrow$ and $\downarrow$) and **inference rule names** that make the following figures correct verifications **or** explain why that is **impossible** to do correctly. As in lecture, you do not need to decorate typing judgments $a : \tau$ with $\uparrow$ or $\downarrow$.

15 **Task 1**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x{:}\tau.\ (A(x)\land B(x))\downarrow}u \qquad \dfrac{}{a:\tau}}{A(a)\land B(a)\downarrow}\forall E}{A(a)\downarrow}\land E_1}{A(a)\ \uparrow}\downarrow\uparrow}{\forall x{:}\tau.\ A(x)\ \uparrow}\forall I^a}{(\forall x{:}\tau.\ (A(x)\land B(x)))\supset(\forall x{:}\tau.\ A(x))\ \uparrow}\supset I^u$$

15 **Task 2**

$$\dfrac{\dfrac{\dfrac{}{\exists x{:}\tau.\ C(x)\ \downarrow}u \qquad \dfrac{}{a:\tau} \quad \dfrac{\dfrac{}{C(a)\ \downarrow}w}{C(a)\ \uparrow}\downarrow\uparrow}{C(a)\ \uparrow}\text{incorrect }\exists E^a}{(\exists x{:}\tau.\ C(x))\supset C(a)\ \uparrow}\supset I^u$$

Solution: This cannot be completed to a verification, because the $\exists E$ rule has been applied incorrectly. The parameter a that it introduces cannot leave its scope, so cannot occur in its conclusion. Indeed, the conclusion is not true, because it does not follow that C is true of a just because it is true of something (e.g., $\tau = \mathsf{nat}$ and $C(x) \equiv (x = 0)$).