Constructive Logic (15-317), Spring 2020
Assignment 5: Sequent Calculus

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Due: Tuesday, February 25, 2020, 11:59 pm

1 Sequent Proofs

In this problem, you will write several sequent calculus proofs. You may silently drop antecedents you no longer need in the remainder of the sequent proof, but beware of dropping antecedents too early!

Task 1 (7 points)**.** Provide a proof of the following sequent:

$$\Longrightarrow A \supset ((B \supset A) \wedge (A \vee C))$$

Task 2 (7 points)**.** Provide a proof of the following sequent:

$$(A \wedge B) \supset C \Longrightarrow A \supset (B \supset C)$$

Task 3 (7 points)**.** Provide a proof of the following sequent:

$$\Longrightarrow ((A \supset \bot) \vee B) \supset (A \supset B)$$

Task 4 (8 points)**.** Provide a proof of the following sequent:

$$\Longrightarrow ((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$$

2 Cut and Natural Deduction

The central theorem of structural proof theory is the closure of sequent calculus under the principle of *cut*; the statement of cut depends on the logic, but for our purposes it can be stated as follows.

Theorem 1 (Cut)**.** If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$.

This theorem can be used to prove many difficult properties about a proof system, including consistency, constructivity, and others. In mathematics, the same technique is also used to establish difficult coherence theorems for higher-dimensional structures.

Another theorem about intuitionistic sequent calculus is closure under the principle of *weakening*, stated as follows.

Theorem 2 (Weakening)**.** If $\Gamma \Longrightarrow C$ then $\Gamma, A \Longrightarrow C$.

Using Theorems 1 and 2, we can prove some results that relate the sequent calculus back to the system of natural deduction that we focused on in the first portion of the class:

Task 5 (5 points)**.** Using Theorem 1 and/or Theorem 2, prove:

If $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.

In particular, you should not use any induction in your argument.

Task 6 (5 points)**.** Using Theorem 1 and/or Theorem 2, prove:

If $\Gamma \Longrightarrow A \supset B$ *and* $\Gamma \Longrightarrow A$*, then* $\Gamma \Longrightarrow B$.

As before, you should not use any induction in your argument.

Note the resemblance of these two theorems to the $\wedge E$ and $\supset E$ rules of natural deduction. Indeed, it is possible to prove similar variants all of the elimination rules of natural deduction admissible in the sequent calculus in much the same way. Likewise, if we slightly modify the rules of natural deduction to make hypotheses explicit, as they are in the sequent calculus, we can translate in the other direction as well.

3 KeYmaera I

Task 7 (1 point)**.** Follow the instructions below to prove $\Longrightarrow A \supset A$ in KeYmaera I.

1. Set up KeYmaera I locally.
2. In your browser of choice, visit `localhost:8090`, and create a new model using the file `hw5_3.kyx`.
3. Apply the `implycR` rule, either by clicking on the arrow in `A -> A`, or from the drop-down menu titled "Propositional".
4. Observe how the goal sequent has changed.
5. Apply the `id` rule from the drop-down menu.
6. Export the proof as `hw5_3_Proof.kyx`.
7. Submit `hw_5_3_Proof.kyx` to autolab.