Constructive Logic (15-317), Spring 2020
Assignment 7:

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Due: Tuesday, April 23, 2020, 11:59 pm

1 Proofs in Linear Logic

Recall that for linear logic we use the judgment $\Gamma; \Delta \Vdash A\ \mathit{true}$, where Γ consists of unrestricted resources $B\ \mathit{ures}$, and Δ consists of resources $C\ \mathit{res}$. We write the empty context as `\cdot`, as in the sequent $\cdot; \Delta \Vdash A\ \mathit{true}$.

If you are not using LaTeX, you can type this judgment as `G ; D ||- A true`, and use a period for the empty context, as in `. ; D ||- A true`.

Task 1 (6)**.** For each of the following sequents, prove that they hold by constructing a sequent calculus proof or explain in terms of resources why they do not hold. For instance, we might note that $\cdot; A\ \mathit{res} \nVdash A \otimes A\ \mathit{true}$ because we only have one A available, but we need two to prove $A \otimes A$.

(1) $\cdot; A \otimes B\ \mathit{res} \Vdash A \mathbin{\&} B\ \mathit{true}$

(2) $\cdot; !(A \mathbin{\&} B)\ \mathit{res} \Vdash !(A) \otimes !(B)\ \mathit{true}$

(3) $\cdot; (A \oplus B) \multimap C\ \mathit{res} \Vdash (A \multimap C) \mathbin{\&} (B \multimap C)\ \mathit{true}$

2 Proof Expansions in Linear Logic

Dual to cut and proof reductions are identity and proof expansions. The admissibility of cut in linear logic tells us that if we can derive $A\ \mathit{true}$ from some assumptions, then those same assumptions entitle us to use the $A\ \mathit{res}$ — that is, true propositions can be used as resources. By contrast, the admissibility of a general identity rule

$$\frac{}{\Gamma; A\ \mathit{res} \Vdash A\ \mathit{true}}\ \mathsf{id}(A)$$

tells us the opposite — a resource $A\ \mathit{res}$ is sufficient to conclude that $A\ \mathit{true}$. Just as we proved cut admissible by an induction whose individual cases were proof reductions, we prove identity admissible by an induction (this time just on the structure of A) whose individual cases are proof expansions. An example expansion is shown below:

$$\frac{}{\Gamma; A \mathbin{\&} B\ \mathit{res} \Vdash A \mathbin{\&} B\ \mathit{true}}\ \mathsf{id}(A \mathbin{\&} B)$$

(1)	$\Gamma; A\ \mathit{res} \Vdash A\ \mathit{true}$	By i.h. on A, as A is smaller than $A \mathbin{\&} B$
(2)	$\Gamma; A \mathbin{\&} B\ \mathit{res} \Vdash A\ \mathit{true}$	By rule $\&L_1$ on (1)
(3)	$\Gamma; B\ \mathit{res} \Vdash B\ \mathit{true}$	By i.h. on B, as B is smaller than $A \mathbin{\&} B$
(4)	$\Gamma; A \mathbin{\&} B\ \mathit{res} \Vdash B\ \mathit{true}$	By rule $\&L_2$ on (3)
(5)	$\Gamma; A \mathbin{\&} B\ \mathit{res} \Vdash A \mathbin{\&} B\ \mathit{true}$	By rule $\&R$ on (2) and (4)

Make sure to clearly state what parameter is smaller than in the current case when using your inductive hypothesis.

Task 2 (3)**.** Complete the following expansion:

$$\dfrac{}{\Gamma; A \multimap B\ \mathit{res} \Vdash A \multimap B\ \mathit{true}}\ \mathsf{id}(A \multimap B)$$

Task 3 (3)**.** Complete the following expansion:

$$\dfrac{}{\Gamma; A \otimes B\ \mathit{res} \Vdash A \otimes B\ \mathit{true}}\ \mathsf{id}(A \otimes B)$$

3 Applications

Blocks World is a class of scenarios in which there is a table, some number of blocks which can be stacked on top of each other, and a robotic arm which can pick up and move blocks. We will briefly look at how to model this situation using linear logic. The following atomic predicates are used:

(1) empty means that the robotic arm's hand is empty.

(2) $\mathsf{holds}(x)$ means that the hand is holding block x.

(3) $\mathsf{clear}(x)$ means that the block x does not have anything on top of it.

(4) $\mathsf{on}(x, y)$ means that the block x is directly on top of the block y.

(5) $\mathsf{on\_table}(x)$ means that the block x is sitting directly on the table.

(6) space means that there is an empty space on the table that can fit a block.

There are four types of possible state transitions in Blocks World:

(a) The hand, if not holding any block, can pick up a block that is on the table and has nothing on top of it, leaving a block-sized space on the table.

(b) The hand, if not holding any block, can pick up the top block of a stack of blocks, exposing the next block down.

(c) The hand, if holding a block, can place it in an empty space on the table.

(d) The hand, if holding a block, can place it on top of an existing stack of blocks.

We can formalize transition (a) as the following axiom in linear logic: [1]

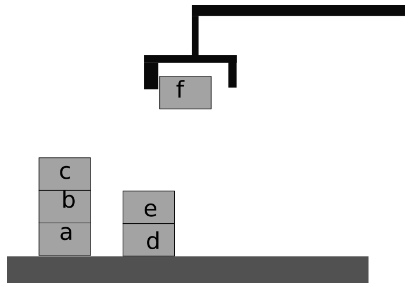
$$!\,(\mathsf{empty} \otimes \mathsf{clear}(x) \otimes \mathsf{on\_table}(x) \multimap \mathsf{holds}(x) \otimes \mathsf{space})$$

Note that we use the exponential ! here to indicate that this is unrestricted, as we may take this action any number of times.

Task 4 (3)**.** Write linear logic axioms describing transitions (b)-(d) from above.

Task 5 (3)**.** Consider the following Blocks World scenario:

[1] Technically, this is an axiom *schema*, and to get an axiom, you need to instantiate all free variables with blocks.



Write a proposition in linear logic which expresses this configuration, assuming that the table can fit **three** blocks total directly on it.

For comparison, the configuration with no blocks would be represented as $\mathsf{empty} \otimes \mathsf{space} \otimes \mathsf{space} \otimes \mathsf{space}$.

Hint: It may help to think about what invariants hold for all configurations, and then to think about what invariants hold for all configurations with the same set of blocks.

Task 6 (3)**.** Write a proposition in linear logic expressing that the blocks are sorted alphabetically in a single stack, with a at the bottom of the stack.

Task 7 (3)**.** Do you think the proposition from task 6 is provable from the axioms and the initial state given in task 5? If so, briefly justify why. If not, briefly justify why not. **You do not need to write a proof.**

Task 8 (5)**.** Suppose we are in a state with two towers of blocks and at least one empty space — something of the form

$$\mathsf{space} \otimes \mathsf{clear}(a) \otimes \mathsf{on}(a, b) \otimes \mathsf{clear}(c) \otimes \mathsf{on}(c, d) \otimes \ldots$$

Write a procedure to swap the top blocks a and b of each tower by listing the sequence of axioms that need to be applied and which block is being moved at each step.

Task 9 (2)**.** How general is your procedure? Describe two different changes to the initial state that would require you to change your procedure, and briefly explain what goes wrong.