Final Exam
15-317/657 Constructive Logic
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Instructions

• For fairness reasons all answers must be typed on a computer in text editors/text-processing (e.g. LaTeX) and submitted as PDF. Besides PDF viewers, no other software is allowed and no handwritten answers/scans are accepted. You can use scratch paper but not hand it in.

• Only the following resources can be used during this exam:
  1. 15317 lecture and recitation notes
  2. editors or text-processing software
  3. private Piazza posts or email with course staff

All other communications with anyone about the exam or this course during the exam period constitute an academic integrity violation.

• You have 24 hours from when the exam was available to complete it.

• There are 4 problems on 5 pages.

• Submit on GradeScope → Final → Submit assignment

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1 Proof Terms (90 points)

This question studies proof terms of natural deduction. Recall that a proof term is called abnormal if it can be reduced by some local reduction of proof terms. Otherwise normal/irreducible.

10 Task 1 Give a normal proof term for \(((A \supset C) \land (B \supset C)) \supset ((A \lor B) \supset (C \lor C))\) or explain why that is impossible.

10 Task 2 Give an abnormal proof term for \((A \supset (B \land C)) \supset (A \supset C)\) or explain why that is impossible.

10 Task 3 Give a normal proof term justifying \(A \supset ((A \lor B) \supset A)\) or explain why that is impossible.

10 Task 4 Give an abnormal proof term justifying \(A \supset ((A \lor B) \supset A)\) or explain why that is impossible.

10 Task 5 Give an abnormal proof term justifying \((A \lor B) \supset A\) or explain why that is impossible.

20 Task 6 Briefly explain whether there is a true proposition \(A\) of intuitionistic propositional logic for which there is no proof term \(M\) such that \(M : A\) proves.

20 Task 7 Briefly explain whether there is a true proposition \(A\) of intuitionistic propositional logic for which there is no abnormal proof term \(M\) such that \(M : A\) proves.
2 Propositional Theorem Proving (80 points)

The contraction-free sequent calculus $\rightarrow$ is sound and complete w.r.t. $\Rightarrow$ and terminates: all its premises are strictly smaller in a well-founded ordering. Each of the following tasks drops one rule from our original contraction-free sequent calculus and replaces it with another. Explain whether these properties still hold when replacing only the indicated rule and mark (s) for sound wrt. $\Rightarrow$, (u) for unsound, (c) for complete wrt. $\Rightarrow$, (i) for incomplete, (t) for terminating, (n) for nonterminating. If they fail, show an example demonstrating the failure.

To get you started here’s a simple example: Replacing rule $\land R$ by rule $P_0$ would make it

$$\begin{align*}
\Gamma &\rightarrow A \\
\Gamma &\rightarrow B \\
\Gamma &\rightarrow A \land B \\
\Gamma &\rightarrow A
\end{align*}$$

$P_0$ because $\rightarrow \top \land \bot$ proves by $P_0 + \top R$ but is (constructively) false as it implies $\bot$ by $\land L$.

(u) because every sequent provable by $\land R$ is provable by $P_0$, which has a subset of the premises of $\land R$.

(t) the same ordering shows termination because $P_0$ produces a subset of the premises of $\land R$.

20 Task 1 Explain what happens when we only replace rule $\lor \supset L$ by rule $P_1$:

$$\begin{align*}
\Gamma, A_1 \supset B, A_2 \supset B &\rightarrow C \\
\Gamma, (A_1 \lor A_2) \supset B &\rightarrow C
\end{align*}$$

$P_1$ because $\rightarrow \top \lor \bot$ proves by $P_0 + \top R$ but is (constructively) false as it implies $\bot$ by $\land L$.

(c) every sequent provable by $\land R$ is provable by $P_0$, which has a subset of the premises of $\land R$.

(n) the same ordering shows termination because $P_0$ produces a subset of the premises of $\land R$.

20 Task 2 Explain what happens when we only replace rule $\lor R_2$ by rule $P_2$:

$$\begin{align*}
\Gamma &\rightarrow B \\
\Gamma &\rightarrow A \lor B
\end{align*}$$

$P_2$ because $\rightarrow \top \lor \bot$ proves by $P_0 + \top R$ but is (constructively) false as it implies $\bot$ by $\land L$.

(i) every sequent provable by $\land R$ is provable by $P_0$, which has a subset of the premises of $\land R$.

(t) the same ordering shows termination because $P_0$ produces a subset of the premises of $\land R$.

20 Task 3 Explain what happens when we only replace rule $\bot \supset L$ by rule $P_3$:

$$\begin{align*}
\Gamma &\rightarrow C \\
\Gamma, \bot \supset B &\rightarrow C
\end{align*}$$

$P_3$ because $\rightarrow \top \lor \bot$ proves by $P_0 + \top R$ but is (constructively) false as it implies $\bot$ by $\land L$.

(u) every sequent provable by $\land R$ is provable by $P_0$, which has a subset of the premises of $\land R$.

(n) the same ordering shows termination because $P_0$ produces a subset of the premises of $\land R$.

20 Task 4 Explain what happens when we only replace rule $P \supset L$ by rule $P_4$:

$$\begin{align*}
P \in \Gamma, B &\rightarrow C \\
\Gamma &\rightarrow P \Gamma, B &\rightarrow C
\end{align*}$$

$P_4$ because $\rightarrow \top \lor \bot$ proves by $P_0 + \top R$ but is (constructively) false as it implies $\bot$ by $\land L$.

(c) every sequent provable by $\land R$ is provable by $P_0$, which has a subset of the premises of $\land R$.

(t) the same ordering shows termination because $P_0$ produces a subset of the premises of $\land R$.
3 Prolog Principles (50 points)

This question studies symbolic computation in Prolog with polynomials in one variable (written \(x\)). Polynomials are represented as a list of integer coefficients, e.g.:

\[ [5,6,7,8] \text{ represents the polynomial } 5 + 6x + 7x^2 + 8x^3 \]

In this question you will define predicates `padd/3`, `pscale/3`, `pmul/3` to compute the representation of polynomials representing polynomial addition, scaling, and multiplication, respectively. For example, the following queries are expected to succeed:

\[
\text{padd}([1,2,3],[5,6],[6,8,3]), \text{pscale}(3,[1,2],[3,6]), \text{pmul}([1,2,3],[5,7],[5,17,29,21]).
\]

Modes describe the intended ways of using a predicate. Mode `+pol` indicates an input argument that needs to be provided satisfying `pol/1`. Mode `-pol` indicates an output argument satisfying `pol/1` that will be computed by the predicate when all inputs are provided, where:

\[
\text{pol}([A|As]) :- \text{integer}(A), \text{pol}(As).
\]

\text{pol}([]).

10 Task 1 Write a Prolog program `padd(+pol,+pol,-pol)` that takes two `pol` representations as inputs in the first and second arguments and produces a `pol` representation of their sum as the output in the third argument.

10 Task 2 Write a Prolog program `pscale(+integer,+pol,-pol)` that takes an integer as input in the first argument, a `pol` representation as input in the second argument and produces a `pol` representation of the second argument multiplied/scaled by the first argument as the output in the third argument.

30 Task 3 Write a Prolog program `pmul(+pol,+pol,-pol)` that takes two `pol` representations as inputs in the first and second arguments and produces a `pol` representation of the product of the input polynomials as the output in the third argument.
4 Linear Logic Cuts (80 points)

This question studies cuts in linear logic. We simply write $\Delta, A \vdash \vdash C$ for $\Delta, A \text{ res} \vdash \vdash C \text{ true}$. Recall that the \textit{linear} cut theorem for linear logic constructs a deduction $\mathcal{F}$ from deductions $\mathcal{D}$ and $\mathcal{E}$ and (just like the ordinary cut theorem for intuitionistic logic) is also proved by induction on the structure of the formula $A$ as well as the deductions $\mathcal{D}$ and $\mathcal{E}$.

\[ \frac{\mathcal{D}}{\mathcal{F}} \quad \frac{\mathcal{E}}{\mathcal{F}} \]

**Theorem 1 (Linear cut)** If $\Delta \vdash A$ and $\Delta', A \vdash C$ then $\Delta, \Delta' \vdash C$.

**Task 1** Provide and briefly explain a counterexample justifying from its resource semantics why the \textit{ordinary} structural cut theorem of intuitionistic logic does \textit{not} hold for linear logic:

If $\Delta \vdash A$ and $\Delta, A \vdash C$ then $\Delta \vdash C$

**Task 2** Commodore Horgiatiki performed one case of linear cut elimination. But he is missing some parts and is unsure whether he got a correct proof. Fill in all missing arguments and justifications and steps so that you obtain a complete proof. If there are any errors or missing justifications in Horgiatiki’s proof, clearly mark and explain in one line. Unnecessary steps are not necessarily incorrect but still need a justification of their (in)correctness.

\[ \mathcal{D} = \frac{D_1 \vdash A_1 \quad D_2 \vdash A_2}{\Delta \vdash A_1 \& A_2} \& R \quad \text{and} \quad \mathcal{E} = \frac{\mathcal{E}_1}{\Delta', A_1 \vdash C} \& L_1 \]

\[ \begin{align*}
\Delta \vdash A_1 & \quad 1 \text{ By} \quad \Delta \vdash A_2 \\
\Delta \vdash A_2 & \quad 2 \text{ By} \\
\Delta', A_1 \vdash C & \quad 3 \text{ By} \\
\Delta', A_2 \vdash C & \quad 4 \text{ By} \\
\Delta, \Delta' \vdash C & \quad 5 \text{ By}
\end{align*} \]

**Task 3** Prove the case of the \textit{linear} cut theorem where $\mathcal{D}$ ends with $\neg\neg R$ and $\mathcal{E}$ ends with $\neg\neg L$:

\[ \mathcal{D} = \frac{\Delta, A_1 \vdash A_2}{\Delta, \vdash A_1 \neg \neg A_2} \neg\neg R \quad \text{and} \quad \mathcal{E} = \frac{\mathcal{E}_1 \quad \mathcal{E}_2}{\Delta_1' \vdash A_1 \quad \Delta_2' \vdash A_2 \vdash C} \neg\neg L \]

**Task 4** When replacing $\neg\neg$ by $\supset$ and $\vdash \vdash$ by $\implies$ does a proof of Task 3 justify the case of cut formula $A_1 \supset A_2$ as principal formula of the ordinary cut theorem for intuitionistic logic? Explain.