Lecture Notes on
Linear Chaining

15-317: Constructive Logic

Frank Pfenning   André Platzer

Lecture 22
April 21, 2020

The previous lecture used the concept of inversion for linear logic to identify which rules can always be used without regret, because they make progress without affecting provability. That leaves only the more difficult connectives with the noninvertible rules to be dealt with. Naively, what could happen is that rule applications zigzag by reaching a hard decision of using a noninvertible rule, but then go off on a tangent to consider other things before checking whether that decision payed off. We will put the concept of chaining to use to make the proof calculus commit to the exploration of the choice of a rule for a connective.

1 Linear Chaining

In light of the admissibility of cut and identity, the truth and resource judgment of linear logic coincide, so we drop them from the sequent notation $\Delta, A \Vdash C$ is short for $\Delta, A\,\mathit{res} \Vdash C\ \mathit{true}$. We now develop the chaining linear logic calculus for focusing sequents $\Delta \mapsto C$ and relate it to $\Delta \Vdash C$.

When you face a question of the form $\Delta \mapsto A \oplus (B \oplus C)$, then your choice is to prove either A or B or C from Δ. But proof search first actually only faces the question of whether to apply $\oplus R_1$ or $\oplus R_2$ rule. In particular, proof search could first use the $\oplus R_2$ rule and face $\Delta \mapsto B \oplus C$ and then look through Δ to see what rules could be applied there. The intuition behind chaining is that, once we consider $A \oplus (B \oplus C)$, we should stick with investigating its resulting choices for proof search without being distracted by anything else. Interestingly, this chaining approach can be used for all noninvertible rules.

Recall *negative* connectives have invertible right rules that can be used immediately. *Positive* connectives have invertible left rules that can be used immediately. As in intuitionistic logic, atomic propositions P have no natural polarity, so could be designated negatively or positively (just never both for the same atomic proposition).

Negative	A^-	$::=$	$A \multimap B \mid A \mathbin{\&} B \mid \top \mid P^-$
Positive	A^+	$::=$	$A \otimes B \mid A \oplus B \mid \mathbf{1} \mid \mathbf{0} \mid !A \mid P^+$
Formulas	A	$::=$	$A^- \mid A^+$

Focusing Constraint: *No sequent in a chaining proof can focus on more than one formula*, either in the antecedent or the succedent.

Antecedents	Δ	$::= \cdot \mid \Delta, A \mid \Delta, [A]$
Succedent	γ	$::= A \mid [A]$
Chaining Sequents	$\Gamma; \Delta \mapsto \gamma$	

Invertible rules stay how they were for linear logic. Noninvertible rules, however, are now only applicable if their respective proposition is in focus. Their focus then inherits to the propositions resulting from the rule.

$$\frac{\Delta, A \mapsto B}{\Delta \mapsto A \multimap B} \multimap R \qquad \frac{\Delta \mapsto [A] \quad \Delta', [B] \mapsto C}{\Delta, \Delta', [A \multimap B] \mapsto C} \multimap L$$

$$\frac{\Delta \mapsto [A] \quad \Delta' \mapsto [B]}{\Delta, \Delta' \mapsto [A \otimes B]} \otimes R \qquad \frac{\Delta, A, B \mapsto \gamma}{\Delta, A \otimes B \mapsto \gamma} \otimes L$$

Observe how the difference between rules with focus and the prior rules without focus is significantly more pronounced in case Δ is a longer list of resources. The focusing rules are in Figure 1.

In particular, we chose to always make invertible rules applicable, rather than paying attention to which left/right inversion phase we currently pursue. While the rules do a good job of keeping a proposition and the propositions resulting from its use in focus, the remaining question is what to focus on in the first place.

$$\frac{\Delta \mapsto [A^+]}{\Delta \mapsto A^+} \text{focus}R \qquad \frac{\Delta, [A^-] \mapsto C}{\Delta, A^- \mapsto C} \text{focus}L$$

Of course, it is imperative that the focusing rules can only be used when no other proposition is already in focus. Furthermore, only positive propositions can be put in focus in the succedent while only negative propositions deserve focus in the antecedent.

Multiplicative Connectives

$$\frac{\Delta \mapsto [A] \quad \Delta' \mapsto [B]}{\Delta, \Delta' \mapsto [A \otimes B]} \otimes R \qquad \frac{\Delta, A, B \mapsto C}{\Delta, A \otimes B \mapsto C} \otimes L$$

$$\frac{}{\cdot \mapsto [\mathbf{1}]} \mathbf{1}R \qquad \frac{\Delta \mapsto C}{\Delta, \mathbf{1} \mapsto C} \mathbf{1}L$$

$$\frac{\Delta, A \mapsto B}{\Delta \mapsto A \multimap B} \multimap R \qquad \frac{\Delta \mapsto [A] \quad \Delta', [B] \mapsto C}{\Delta, \Delta', [A \multimap B] \mapsto C} \multimap L$$

Additive Connectives

$$\frac{\Delta \mapsto A \quad \Delta \mapsto B}{\Delta \mapsto A \,\&\, B} \&R \qquad \frac{\Delta, [A] \mapsto C}{\Delta, [A \,\&\, B] \mapsto C} \&L_1 \qquad \frac{\Delta, [B] \mapsto C}{\Delta, [A \,\&\, B] \mapsto C} \&L_2$$

$$\frac{}{\Delta \mapsto \top} \top R \qquad \text{no } \top L \text{ rule}$$

$$\frac{\Delta \mapsto [A]}{\Delta \mapsto [A \oplus B]} \oplus R_1 \qquad \frac{\Delta \mapsto [B]}{\Delta \mapsto [A \oplus B]} \oplus R_2 \qquad \frac{\Delta, A \mapsto C \quad \Delta, B \mapsto C}{\Delta, A \oplus B \mapsto C} \oplus L$$

$$\text{no } \mathbf{0}R \text{ rule} \qquad \frac{}{\Delta, \mathbf{0} \mapsto C} \mathbf{0}L$$

Figure 1: Chaining in intuitionistic Linear Logic

The focus can only be lost when the polarity of the formula in focus no longer requires chaining, because the polarity indicates an invertible rule:

$$\frac{\Delta \mapsto A^-}{\Delta \mapsto [A^-]}\ \mathrm{blur}R \qquad \frac{\Delta, A^+ \mapsto C}{\Delta, [A^+] \mapsto C}\ \mathrm{blur}L$$

This leaves only the atomic propositions to be worried about. Depending on their polarity, they can lose focus by blurring or have to remain in focus. For example, $\Delta \mapsto [P^+ \otimes Q^+]$ requires a proof of $\Delta_1 \mapsto [P^+]$ and a proof of $\Delta_2 \mapsto [Q^+]$ for some division $\Delta = (\Delta_1, \Delta_2)$ of the antecedent resources. Unfortunately, losing focus is disallowed, because P^+ has the wrong polarity for blur$R$, and no rule could decompose an atomic proposition P^+ any further. Andreoli's insight [1] is that the proof then completes if the resources exactly match the required atomic proposition and fails in all other cases:

$$\frac{}{P^+ \mapsto [P^+]}\ \mathrm{id}_{P^+} \qquad \frac{}{[P^-] \mapsto P^-}\ \mathrm{id}_{P^-}$$

Soundness of these rules is mostly obvious even if completeness is not at all obvious in the slightest bit.

2 Example: Backwards Negativity

Consider the following example with atomic propositions a, b, c:

$$a, b, a \multimap (b \multimap c), c \multimap d \Vdash d$$

Without focusing, this sequent has a number of different proofs, including even more failed proof attempts. In order to narrow down the proof search, we use the chaining calculus and need to decide on a polarity for the atoms. It turns out this particular example has exactly one proof, no matter what polarity is chosen. Let's make all atoms negative:

$$a^-, b^-, a^- \multimap (b^- \multimap c^-), c^- \multimap d^- \mapsto d^-$$

The first step is to focus because no invertible connectives remain. Focusing on the negative atomic propositions on the left will fail quickly for lack of applicable rules since the right-hand side does not match:

$$..., [a^-] \Vdash d^-$$

If we were to focus on the first implication, the proof would not proceed much further:

$$\dfrac{\dots \mapsto [a^-] \qquad \dfrac{\dots \mapsto [b^-] \qquad \dots, [c^-] \mapsto d^-}{\dots, [b^- \multimap c^-] \mapsto d^-} \multimap L}{\dots, [a^- \multimap (b^- \multimap c^-)] \mapsto d^-} \multimap L$$

Again, the proof attempt fails independently of the partition of the resources ..., because no rule applies to $[c^-] \mapsto d^-$ when $c \neq d$, and no other propositions are even under consideration when c^- is in focus. Moreover, no other rules were applicable along the way.

Consequently, the only possible focus is on the last implication, because only its conclusion matches the desired succedent d^-:

$$\dfrac{\dfrac{\dfrac{a^-, b^-, a^- \multimap (b^- \multimap c^-), \mapsto c^-}{a^-, b^-, a^- \multimap (b^- \multimap c^-), \mapsto [c^-]} \text{blur}R \qquad \dfrac{}{[d^-] \mapsto d^-} \text{id}_{d^-}}{a^-, b^-, a^- \multimap (b^- \multimap c^-), [c^- \multimap d^-] \mapsto d^-} \multimap L}{a^-, b^-, a^- \multimap (b^- \multimap c^-), c^- \multimap d^- \mapsto d^-} \text{focus}L$$

This is the only successful proof because no other rules were applicable or they fail quickly if one tries. The remaining subgoal also only proves in exactly one way by focusing on its implication, which ultimately proves the succedent c^-. It is quite remarkable how only a single proof succeeds and other proof attempts are suppressed quickly. For this proof to succeed, we did have to be clever about partitioning resources appropriately. On a sheet of paper, this is manageable, but a theorem prover needs more explicit management to cut back on proof search.

Just like in intuitionistic chaining, observe how the proofs resulting from all negative atoms perform goal-directed alias backward chaining proofs.

3 Example: Forward Positivity

Now let's make all atomic propositions in the same example positive:

$$a^+, b^+, a^+ \multimap (b^+ \multimap c^+), c^+ \multimap d^+ \mapsto d^+$$

Focusing on the succedent d^+ fails, because d^+ is not readily available as a resource yet. Focusing on the second implication should not work:

$$\dfrac{a^+, b^+, a^+ \multimap (b^+ \multimap c^+) \mapsto [c^+] \quad [d^+] \mapsto d^+}{a^+, b^+, a^+ \multimap (b^+ \multimap c^+), [c^+ \multimap d^+] \mapsto d^+} \multimap L$$

Since c^+ is not available as a resource, the first premise has no applicable rule. Focusing on the atoms a^+ or b^+ in the antecedent would be forbidden, because they have the wrong polarity for focus$L$. Consequently, we have to focus on the first implication:

$$\dfrac{\dfrac{\dfrac{}{a^+ \mapsto [a^+]}\,\text{id}_{a^+} \quad \dfrac{\dfrac{}{b^+ \mapsto [b^+]}\,\text{id}_{b^+} \quad \dfrac{c^+, c^+ \multimap d^+ \mapsto d^+}{[c^+], c^+ \multimap d^+ \mapsto d^+}\,\text{blur}L}{b^+, [b^+ \multimap c^+], c^+ \multimap d^+ \mapsto d^+}\multimap L}{a^+, b^+, [a^+ \multimap (b^+ \multimap c^+)], c^+ \multimap d^+ \mapsto d^+}\multimap L}{a^+, b^+, a^+ \multimap (b^+ \multimap c^+), c^+ \multimap d^+ \mapsto d^+}\,\text{focus}L$$

The remaining premise proves by focusing on the implication. Observe how positive polarity leads to a forward chaining proof that mostly ignores the succedent and, instead, derives more and more resources in the antecedent.

4 Soundness of Chaining

Chaining is sound for linear logic, that is, every sequent provable in the chaining calculus is also provable in the previous linear logic sequent calculus. Soundness is mostly obvious, because the chaining calculus merely imposed more restrictions on the deduction by limiting which rules are applicable, not really changing the rules.

Theorem 1 (Soundness) *If* $\Delta \mapsto A$ *then* $\Delta \Vdash A$.

Proof: By straightforward induction on the structure of the deduction proving $\Delta \mapsto A$ by eliding the focus information and skipping blur/focus proof steps. □

5 Completeness of Chaining

The significantly harder direction for restrictions of proof calculi is completeness, i.e., that every sequent provable previously is also provable with chaining. Since we don't know any better, let's simply try a direct inductive proof and then try and patch it up.

Theorem 2 (Completeness) *If* $\Delta \Vdash A$ *then* $\Delta \mapsto A$.

Proof: Attempting induction on the structure of the deduction $\Delta \overset{\mathcal{D}}{\Vdash} A$. First come the easy cases:

Case:

$$\mathcal{D} = \frac{}{P \Vdash P}\ \mathsf{id}$$

Subcase: $P = P^+$ has positive polarity

$$\dfrac{\dfrac{}{P^+ \mapsto [P^+]}\ \mathsf{id}_{P^+}}{P^+ \mapsto P^+}\ \text{focus}R$$

Subcase: $P = P^-$ has negative polarity

$$\dfrac{\dfrac{}{[P^-] \mapsto P^-}\ \mathsf{id}_{P^-}}{P^- \mapsto P^-}\ \text{focus}L$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Delta, A \Vdash B\end{array}}{\Delta \Vdash A \multimap B}\ {\multimap}R$$

This is provable with the premise provable by induction hypothesis

$$\dfrac{\begin{array}{c}\mathrm{IH}(\mathcal{D}_1) \\ \Delta, A \mapsto B\end{array}}{\Delta \mapsto A \multimap B}\ {\multimap}R$$

Case:

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ \Delta \Vdash A & \Delta', B \Vdash C\end{matrix}}{\Delta, \Delta', A \multimap B \Vdash C} \multimap L$$

The induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$ proves

$$\Delta \mapsto A \qquad \Delta', B \mapsto C$$

In order to prove the conjecture, the canonical way forward would be to focus on using $A \multimap B$:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ \Delta \Vdash [A] & \Delta', [B] \Vdash C\end{matrix}}{\Delta, \Delta', [A \multimap B] \mapsto C} \multimap L}{\Delta, \Delta', A \multimap B \mapsto C} \text{focus}L$$

Now, unfortunately, the induction hypothesis does not imply that these two resulting sequents are provable, because of their different focus. Even if $\Delta \Vdash A$ implies $\Delta \mapsto A$ by induction hypothesis, since the former as a smaller proof, that does not imply $\Delta \mapsto [A]$, because this may have been the wrong way of focusing for the proof to succeed. Indeed, there is a direct counterexample. The following proves by focusing on $a^+ \multimap b^+$ but does not prove after focusing on the succedent, because no rule applies:

$$a^+, a^+ \multimap b^+ \mapsto b^+$$

Instead, if we prove that the chaining calculus for linear logic admits cuts, then the differences can be ironed out as follows:

$$\dfrac{\dfrac{\begin{matrix}\text{IH}(\mathcal{D}_1) & \\ \Delta \mapsto A & A, A \multimap B \mapsto B\end{matrix}}{\Delta, A \multimap B \mapsto B} \text{cut}(A) \quad \begin{matrix}\text{IH}(\mathcal{D}_2) \\ \Delta', B \mapsto C\end{matrix}}{\Delta, \Delta', A \multimap B \mapsto C} \text{cut}(B)$$

The middle premise completes if identity is admissible in the chaining calculus:

$$\dfrac{\dfrac{\dfrac{}{A \mapsto [A]} \text{id} \quad \dfrac{}{[B] \mapsto B} \text{id}}{A, [A \multimap B] \mapsto B} \multimap L}{A, A \multimap B \mapsto B} \text{focus}L$$

□

Consequently, completeness comes down to the admissibility of identity and cut.

Theorem 3 (Identity) *The identity rules are admissible for chaining linear logic:*

$$\frac{}{A \mapsto [A]}\ \mathsf{id}_{A[]} \qquad \frac{}{[A] \mapsto A}\ \mathsf{id}_{[]A} \qquad \frac{}{A \mapsto A}\ \mathsf{id}_{A}$$

Proof: By mutual induction on the structure of A where A is considered smaller than $[A]$. □

The proof of the cut theorem is the primary difficulty owing to its need of maintaining the focusing restriction that only ever one proposition is in focus in any sequent.

Theorem 4 (Cut) *The following four cut rules are admissible:*

$$\frac{\Delta \mapsto A^- \quad \Delta', A^- \mapsto \gamma}{\Delta, \Delta' \mapsto \gamma}\ cut_{A^-} \qquad \frac{\Delta \mapsto A^+ \quad \Delta', A^+ \mapsto C}{\Delta, \Delta' \mapsto C}\ cut_{A^+}$$

$$\frac{\Delta \mapsto [A] \quad \Delta', A \mapsto \gamma}{\Delta, \Delta' \mapsto \gamma}\ cut[]_A \qquad \frac{\Delta \mapsto A \quad \Delta', [A] \mapsto C}{\Delta, \Delta' \mapsto C}\ cut_A[]$$

We will not pursue its nontrivial proof here.

6 Persistent Resources

The exponential connective of linear logic leads to the need to handle unlimited/persistent resources, which can be added easily to the judgmental formulation of linear logic. Persistent resources are never in focus but become the focus when copying a persistent resource. Consequently, we use the name focus for the copy rule to better explain its role. To make sure still that no positive atomic proposition ends up in focus accidentally, the focusing copy rule does not apply for positive atomic propositions:

$$\frac{\Gamma, A; \Delta, [A] \mapsto C \quad A \text{ is no } P^+}{\Gamma, A; \Delta \mapsto C}\ \text{focus!}$$

Like all other focus rules, this rule can only be used if no proposition is presently in focus to maintain the invariant that at most one proposition has the focus. The identity rule is duplicated for positive atomic propositions:

$$\frac{}{\Gamma, P^+; \cdot \mapsto [P^+]}\ \text{id}!_{P^+}$$

The rules for exponential connectives now reduce to unlimited resources.

$$\frac{\Gamma; \cdot \mapsto A}{\Gamma; \cdot \mapsto [!A]}\ !R \qquad \frac{\Gamma, A; \Delta \mapsto C}{\Gamma; \Delta, !A \mapsto C}\ !L$$

What is fairly surprising but important is that the focus is lost in the premise of $!R$.

References

[1] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.