Lecture Notes on Cut Elimination

15-317: Constructive Logic
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1 Introduction

The identity rule of the sequent calculus exhibits one connection between the judgments A *left* and A *right*: If we assume A *left* we can prove A *right*. In other words, the left rules of the sequent calculus are strong enough so that we can reconstitute a proof of A from the assumption A. So the identity theorem (see Section 5) is a global version of the local completeness property for the elimination rules.

The cut theorem of the sequent calculus expresses the opposite: if we have a proof of A *right* we are licensed to assume A *left*. This can be interpreted as saying the left rules are not too strong: whatever we can do with the antecedent A *left* can also be deduced without that, if we know A *right*. Because A *right* occurs only as a succedent, and A *left* only as an antecedent, we must formulate this in a somewhat roundabout manner: If $\Gamma \Longrightarrow A\ \textit{right}$ and $\Gamma, A\ \textit{left} \Longrightarrow J$ then $\Gamma \Longrightarrow J$. In the sequent calculus for pure intuitionistic logic, the only conclusion judgment we are considering is C *right*, so we specialize the above property.

Because it is very easy to go back and forth between sequent calculus deductions of A *right* and verifications of $A\!\uparrow$, we can use the cut theorem to show that every true proposition has a verification, which establishes a fundamental, global connection between truth and verifications. While the sequent calculus is a convenient intermediary (and was conceived as such

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by Gentzen [Gen35]), this theorem can also be established directly using verifications.

2 Admissibility of Cut

The cut theorem is one of the most fundamental properties of logic. Because of its central role, we will spend some time on its proof. In lecture we developed the proof and the required induction principle incrementally to explain its intuition and design principles; here we present the final result as is customary in mathematics. The proof is amenable to formalization in a logical framework; details can be found in a paper by the instructor [Pfe00]. A rule is *admissible* if everything that it proves can be proved without it.

Theorem 1 (Cut) *If* $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$*.*

Proof: By nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut formula, or with an identical cut formula and two derivations, one of which is strictly smaller while the other stays the same. The proof is constructive, which means we show how to transform

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \Longrightarrow C\end{array} \quad \text{to} \quad \begin{array}{c}\mathcal{F}\\ \Gamma \Longrightarrow C\end{array}$$

The proof is divided into several classes of cases. More than one case may be applicable, which means that the algorithm for constructing the derivation of $\Gamma \Longrightarrow C$ from the two given derivations is non-deterministic.

Case: $\mathcal{D}$ is an initial sequent, $\mathcal{E}$ is arbitrary.

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\,\mathsf{id} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma', A, A \Longrightarrow C\end{array}$$

$\Gamma = (\Gamma', A)$	This case
$\Gamma', A, A \Longrightarrow C$	Deduction $\mathcal{E}$
$\Gamma', A \Longrightarrow C$	By Contraction (see Lecture 9)
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', A)$

Case: $\mathcal{D}$ is arbitrary and $\mathcal{E}$ is an initial sequent using the cut formula.

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \frac{}{\Gamma, A \Longrightarrow A}\,\mathsf{id}$$

$A = C$	This case
$\Gamma \Longrightarrow A$	Deduction $\mathcal{D}$

Case: $\mathcal{E}$ is an initial sequent *not* using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma', C, A \Longrightarrow C}\ \mathsf{id}$$

$\Gamma = (\Gamma', C)$	This case
$\Gamma', C \Longrightarrow C$	By rule id
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', C)$

In the next set of cases, the cut formula is the principal formula of the final inference in both $\mathcal{D}$ and $\mathcal{E}$. We only show two of these cases.

Case:

$$\mathcal{D} = \frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \Longrightarrow A_1 & \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2}\ \wedge R$$

$$\text{and} \qquad \mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C}\ \wedge L_1$$

$A = A_1 \wedge A_2$	This case
$\Gamma, A_1 \Longrightarrow C$	By i.h. on $A_1 \wedge A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C$	By i.h. on A_1, $\mathcal{D}_1$, and previous line

Actually we have ignored a detail: in the first appeal to the induction hypothesis, $\mathcal{E}_1$ has an additional hypothesis, A_1, and therefore does not match the statement of the theorem precisely. However, we can always weaken $\mathcal{D}$ to include this additional hypothesis without changing the structure of $\mathcal{D}$ (see the Weakening Theorem in Lecture 9) and then appeal to the induction hypothesis. We will not be explicit about these trivial weakening steps in the remaining cases.

It is crucial for a well-founded induction that $\mathcal{E}_1$ is smaller than $\mathcal{E}$, so even if the same cut formula and same $\mathcal{D}$ is used, $\mathcal{E}_1$ got smaller. Note that we cannot directly appeal to induction hypothesis on A_1,$\mathcal{D}_1$ and $\mathcal{E}_1$ because the additional formula $A_1 \wedge A_2$ might still be used in $\mathcal{E}_1$, e.g., by a subsequent use of $\wedge L_2$.

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2} \supset\! R$$

$$\text{and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C} \supset\! L$$

$A = A_1 \supset A_2$	This case
$\Gamma \Longrightarrow A_1$	By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow A_2$	By i.h. on A_1 from above and $\mathcal{D}_2$
$\Gamma, A_2 \Longrightarrow C$	By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C$	By i.h. on A_2 from above

Note that the proof constituents of the last step $\Gamma \Longrightarrow C$ may be longer than the original deductions $\mathcal{D}, \mathcal{E}$. Hence, it is crucial for a well-founded induction that the cut formula A_2 is smaller than $A_1 \supset A_2$.

Finally note the resemblance of these principal cases to the local soundness reductions in harmony arguments for natural deduction.

In the next set of cases, the principal formula in the last inference in $\mathcal{D}$ is *not* the cut formula. We sometimes call such formulas *side formulas* of the cut, because they play a side role compared to the rule's principal formulas.

Case: If $\mathcal{D}$ ended with an $\wedge L_1$:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma', B_1 \wedge B_2, B_1 \Longrightarrow A\end{array}}{\Gamma', B_1 \wedge B_2 \Longrightarrow A} \wedge L_1 \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Gamma', B_1 \wedge B_2, A \Longrightarrow C\end{array}$$

$\Gamma = (\Gamma', B_1 \wedge B_2)$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_1$ and $\mathcal{E}$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge L_1$
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', B_1 \wedge B_2)$

Case: If $\mathcal{D}$ ended with $\supset L$:

$$\mathcal{D} = \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma', B_1 \supset B_2 \Longrightarrow B_1 & \Gamma', B_1 \supset B_2, B_2 \Longrightarrow A\end{array}}{\Gamma', B_1 \supset B_2 \Longrightarrow A} \supset L$$

$\Gamma = (\Gamma', B_1 \supset B_2)$	This case
$\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_2$ and $\mathcal{E}$
$\Gamma', B_1 \supset B_2 \Longrightarrow C$	By rule $\supset L$ on $\mathcal{D}_1$ and above
$\Gamma \Longrightarrow C$	Since $\Gamma = (\Gamma', B_1 \supset B_2)$

In the final set of cases, A is not the principal formula of the last inference in $\mathcal{E}$. This *overlaps* with the previous cases since A may not be principal on either side. In this case, we appeal to the induction hypothesis on the subderivations of $\mathcal{E}$ and directly infer the conclusion from the results.

Case: If $\mathcal{E}$ ended with $\wedge R$:

$$\begin{array}{c}\mathcal{D} \\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \dfrac{\begin{array}{cc}\mathcal{E}_1 & \mathcal{E}_2 \\ \Gamma, A \Longrightarrow C_1 & \Gamma, A \Longrightarrow C_2\end{array}}{\Gamma, A \Longrightarrow C_1 \wedge C_2} \wedge R$$

$C = C_1 \wedge C_2$	This case
$\Gamma \Longrightarrow C_1$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_2$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C_1 \wedge C_2$	By rule $\wedge R$ on above

Case: If $\mathcal{E}$ ended with $\wedge L_1$:

$$\begin{array}{c}\mathcal{D} \\ \Gamma \Longrightarrow A\end{array} \quad \text{and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma', B_1 \wedge B_2, B_1, A \Longrightarrow C\end{array}}{\Gamma', B_1 \wedge B_2, A \Longrightarrow C} \wedge L_1$$

$\Gamma = (\Gamma', B_1 \wedge B_2)$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge L_1$ from above

□

3 Applications of Cut Admissibility

The admissibility of cut, together with the admissibility of identity (see Section 5), complete our program to find global versions of local soundness and completeness. This has many positive consequences. We already have seen that the sequent calculus (without cut!) must be consistent, because there is no sequent proof of $\bot$.

If we can translate from arbitrary natural deductions to the sequent calculus, then this also means that natural deduction is consistent, and similarly for other properties such as the disjunction property. Once we have the admissibility of cut, the translation from natural deduction to sequent calculus is surprisingly simple. Note that this is somewhat different from the previous translation that worked on *verifications*: here we are interested in translating arbitrary natural deductions.

Theorem 2 *If* $\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\ \mathit{true}\end{array}$ *in natural deduction then* $\Gamma \Longrightarrow A$ *in sequent calculus.*

Proof: By induction on the structure of $\mathcal{D}$. For deductions $\mathcal{D}$ ending in introduction rules, we just replay the corresponding right rule. For example:

Case: $\mathcal{D} = \dfrac{\begin{array}{c}\Gamma \quad \overline{A_1\ \mathit{true}}^{\,u}\\ \mathcal{D}_2\\ A_2\ \mathit{true}\end{array}}{A_1 \supset A_2\ \mathit{true}} \supset\! I^u$

$\Gamma, A_1 \Longrightarrow A_2$ By i.h. on $\mathcal{D}_2$

$\Gamma \Longrightarrow A_1 \supset A_2$ By rule $\supset\! R$

For uses of hypotheses, we fill in a use of the identity rule.

Case: $\mathcal{D} = \overline{A}^{\,u}$

$\Gamma, A \Longrightarrow A$ By id

Finally, the tricky cases: elimination rules. In these cases we appeal to the induction hypothesis wherever possible and then use the admissibility of cut!

Case: $\mathcal{D} = \dfrac{\begin{array}{c}\Gamma \\ \mathcal{D}_1 \\ B \supset A\ \mathit{true}\end{array} \quad \begin{array}{c}\Gamma \\ \mathcal{D}_2 \\ B\ \mathit{true}\end{array}}{A\ \mathit{true}} \supset E$

$\mathcal{E}_1$ proves $\Gamma \Longrightarrow B \supset A$	By i.h. on $\mathcal{D}_1$
$\mathcal{E}_2$ proves $\Gamma \Longrightarrow B$	By i.h. on $\mathcal{D}_2$
To show: $\Gamma \Longrightarrow A$	

At this point we realize that the sequent rules "go in the wrong direction" for this translation. They are designed to let us prove sequents, rather than take advantage of knowledge, such as $\Gamma \Longrightarrow B \supset A$.

However, using the admissibility of cut, we can piece together a deduction of A. First we prove (omitting some redundant antecedents):

$$\mathcal{F} = \dfrac{\dfrac{}{B \Longrightarrow B}\,\mathsf{id} \quad \dfrac{}{A \Longrightarrow A}\,\mathsf{id}}{B \supset A, B \Longrightarrow A} \supset L$$

Then (leaving some trivial instances of weakening implicit):

$\mathcal{F}_1$ proves $\Gamma, B \Longrightarrow A$	By adm. of cut on $\mathcal{E}_1$ and $\mathcal{F}$
$\Gamma \Longrightarrow A$	By adm. of cut on $\mathcal{E}_2$ and $\mathcal{F}_1$

where the last line is what we needed to show.

□

The translation from sequent proofs to verifications is quite straightforward, so we omit it here. But chaining these proof translations together we find that every true propositions A (as defined by natural deduction) has a verification. This closes the loop on our understanding of the connections between natural deductions, sequent proofs, and verifications elegantly.

4 Cut Elimination[1]

Gentzen's original presentation of the sequent calculus included an inference rule for cut. We write $\Gamma \overset{\mathsf{cut}}{\Longrightarrow} A$ for this system, which is just like

[1]This material not covered in lecture

$\Gamma \Longrightarrow A$, with the additional rule

$$\frac{\Gamma \overset{\mathsf{cut}}{\Longrightarrow} A \quad \Gamma, A \overset{\mathsf{cut}}{\Longrightarrow} C}{\Gamma \overset{\mathsf{cut}}{\Longrightarrow} C}\ \mathsf{cut}$$

The advantage of this calculus is that it more directly corresponds to natural deduction in its full generality, rather than verifications, because just like in natural deduction, the cut rule makes it possible to prove an arbitrary other A from the available assumptions Γ (left premise) and then use that A as an additional assumption in the rest of the proof (right premise). The disadvantage is that it cannot easily be seen as capturing the meaning of the connectives by inference rules, because with the rule of cut the meaning of C might depend on the meaning of any other proposition A (possibly even including C as a subformula).

In order to clearly distinguish between the two kinds of calculi, the one we presented is sometimes called the *cut-free sequent calculus*, while Gentzen's calculus would be a *sequent calculus with cut*. The theorem connecting the two is called *cut elimination*: for any deduction in the sequent calculus with cut, there exists a cut-free deduction of the same sequent. The proof is a straightforward induction on the structure of the deduction, appealing to the cut theorem in one crucial place.

Theorem 3 (Cut Elimination) *If $\mathcal{D}$ is a deduction of $\Gamma \overset{\mathsf{cut}}{\Longrightarrow} C$ possibly using the* cut *rule, then there exists a cut-free deduction $\mathcal{D}'$ of $\Gamma \overset{\mathsf{cut}}{\Longrightarrow} C$.*

Proof: By induction on the structure of $\mathcal{D}$. In each case, we appeal to the induction hypothesis on all premises and then apply the same rule to the result. The only interesting case is when a cut rule is encountered.

Case:

$$\mathcal{D} = \frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \overset{\mathsf{cut}}{\Longrightarrow} A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Gamma, A \overset{\mathsf{cut}}{\Longrightarrow} C\end{matrix}}{\Gamma \overset{\mathsf{cut}}{\Longrightarrow} C}\ \mathsf{cut}$$

$\Gamma \Longrightarrow A$ without cut	By i.h. on $\mathcal{D}_1$
$\Gamma, A \Longrightarrow C$ without cut	By i.h. on $\mathcal{D}_2$
$\Gamma \Longrightarrow C$	By the Cut Theorem

□

5 Identity[2]

We permit the identity rule for all propositions. However, the version of this rule for just atomic propositions P is strong enough. We write $\Gamma \stackrel{\mathsf{id}}{\Longrightarrow} A$ for this restricted system In this restricted system, the rule for arbitrary propositions A is *admissible*, that is, each instance of the rule can be deduced. We call this the *identity theorem* because it shows that from an assumption A we can prove the identical conclusion A.

Theorem 4 (Identity) *For any proposition A, we have $A \stackrel{\mathsf{id}}{\Longrightarrow} A$.*

Proof: By induction on the structure of A. We show several representative cases and leave the remaining ones to the reader.

Case: $A = P$ for an atomic proposition P. Then

$$\frac{}{P \stackrel{\mathsf{id}}{\Longrightarrow} P}\ \mathsf{id}$$

Case: $A = A_1 \wedge A_2$. Then

$$\dfrac{\dfrac{\begin{array}{c}\text{By i.h. on } A_1 \text{ and weakening}\\ A_1 \wedge A_2, A_1 \stackrel{\mathsf{id}}{\Longrightarrow} A_1\end{array}}{A_1 \wedge A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_1}\ \wedge L_1 \qquad \dfrac{\begin{array}{c}\text{By i.h. on } A_2 \text{ and weakening}\\ A_1 \wedge A_2, A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_2\end{array}}{A_1 \wedge A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_2}\ \wedge L_2}{A_1 \wedge A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_1 \wedge A_2}\ \wedge R$$

Case: $A = A_1 \supset A_2$. Then

$$\dfrac{\dfrac{\begin{array}{c}\text{By i.h. on } A_1 \text{ and weakening}\\ A_1 \supset A_2, A_1 \stackrel{\mathsf{id}}{\Longrightarrow} A_1\end{array} \qquad \begin{array}{c}\text{By i.h. on } A_2 \text{ and weakening}\\ A_1 \supset A_2, A_1, A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_2\end{array}}{A_1 \supset A_2, A_1 \stackrel{\mathsf{id}}{\Longrightarrow} A_2}\ \supset L}{A_1 \supset A_2 \stackrel{\mathsf{id}}{\Longrightarrow} A_1 \supset A_2}\ \supset R$$

Case: $A = \perp$. Then

$$\frac{}{\perp \stackrel{\mathsf{id}}{\Longrightarrow} \perp}\ \perp L$$

[2]this section not covered in lecture

□

The identity theorem is the global version of the local completeness property for each individual connective. Local completeness shows that a connective can be re-verified from a proof that gives us license to use it, which directly corresponds to $A \overset{\mathsf{id}}{\Longrightarrow} A$. One can recognize the local expansion as embodied in each case of the inductive proof of identity.

References

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