Constructive Logic (15-317), Fall 2016
Recitation 7: Playing with proof systems

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1 All the sequent calculi

We have seen in lecture four different sequent calculi, each improving on the previous for automatic (and, let's be honest, manual) proof search.

1.1 Sequent calculus

First there was sequent calculus, which can be obtained quite straightforwardly from the natural deduction calculus with verification judgments.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C} \supset L$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \Longrightarrow P} \mathit{init} \qquad \frac{}{\Gamma \Longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \Longrightarrow C} \bot L$$

1.2 Restricted sequent calculus

We quickly realize that the sequent calculus above can't be good for proof search, as it keeps a copy of every formula potentially wasting memory and increasing the search space. So we notice we can restrict it and, in the end, the only formula we actually need to keep copies of are implications on the left.

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \longrightarrow P} \mathit{init} \qquad \frac{}{\Gamma \longrightarrow \top} \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C} \bot L$$

1.3 Inversion sequent calculus

Playing around with the calculus above, we notice that some rules are *invertible*, meaning that their premises are justified from the conclusion[1]. Therefore we can eagerly apply those rules when doing proof search, without looking back. This reduces the search space considerably, since we don't need to backtrack on every rule application, only on the non-invertible ones.

$$\frac{\Gamma^-;\Omega,A \xrightarrow{R} B}{\Gamma^-;\Omega \xrightarrow{R} A \supset B}\supset\!R \qquad \frac{\Gamma^-,A\supset B;\cdot \xrightarrow{R} A \quad \Gamma^-;B \xrightarrow{L} C^+}{\Gamma^-,A\supset B;\cdot \xrightarrow{L} C^+}\supset\!L$$

$$\frac{\Gamma^-;\Omega \xrightarrow{R} A \quad \Gamma^-;\Omega \xrightarrow{R} B}{\Gamma^-;\Omega \xrightarrow{R} A\wedge B}\wedge R \qquad \frac{\Gamma^-;\Omega,A,B \xrightarrow{L} C^+}{\Gamma^-;\Omega,A\wedge B \xrightarrow{L} C^+}\wedge L$$

$$\frac{\Gamma^-;\cdot \xrightarrow{R} A}{\Gamma^-;\cdot \xrightarrow{L} A\vee B}\vee R_1 \qquad \frac{\Gamma^-;\cdot \xrightarrow{R} B}{\Gamma^-;\cdot \xrightarrow{L} A\vee B}\vee R_2 \qquad \frac{\Gamma^-;\Omega,A \xrightarrow{L} C^+ \quad \Gamma^-;\Omega,B \xrightarrow{L} C^+}{\Gamma^-;\Omega,A\vee B \xrightarrow{L} C^+}\vee L$$

$$\frac{P\in\Gamma^-}{\Gamma^-;\Omega \xrightarrow{R} P}\,\mathsf{init} \qquad \frac{P = C^+}{\Gamma^-;\Omega,P \xrightarrow{L} C^+}\,\mathsf{init} \qquad \frac{}{\Gamma^-;\Omega \Longrightarrow \top}\top R \qquad \frac{}{\Gamma^-;\Omega,\bot \xrightarrow{L} C^+}\bot L$$

$$\frac{P\notin\Gamma^- \quad \Gamma^-;\Omega \xrightarrow{L} P}{\Gamma^-;\Omega \xrightarrow{R} P}\,\mathsf{LR}_P \qquad \frac{\Gamma^-;\Omega \xrightarrow{L} A\vee B}{\Gamma^-;\Omega \xrightarrow{R} A\vee B}\,\mathsf{LR}_\vee \qquad \frac{\Gamma^-;\Omega \xrightarrow{L} \bot}{\Gamma^-;\Omega \xrightarrow{R} \bot}\,\mathsf{LR}_\bot$$

$$\frac{\Gamma^-;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega,\top \xrightarrow{L} C^+}\top L \qquad \frac{\Gamma^-,P;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega,P \xrightarrow{L} C^+}\,\mathsf{shift}_P \qquad \frac{\Gamma^-,A\supset B;\Omega \xrightarrow{L} C^+}{\Gamma^-;\Omega,A\supset B \xrightarrow{L} C^+}\,\mathsf{shift}_\supset$$

1.4 Contraction-free sequent calculus (a.k.a. G4ip)

Still we have the problem of needing to keep implications on the left around. By analyzing what might happen on the left side of an implication more carefully, we can come up with a calculus where this implicit contraction of implications no longer occurs. This is perfect for proof search and it gives directly a decision procedure for propositional intuitionistic logic (which is good anyway, since this is indeed a decidable fragment).

$$\frac{\Gamma,A \longrightarrow B}{\Gamma \longrightarrow A\supset B}\supset\!R \qquad \frac{P\in\Gamma \quad \Gamma,B\longrightarrow C}{\Gamma,P\supset B\longrightarrow C}P\!\supset\!L \qquad \frac{\Gamma,B\longrightarrow C}{\Gamma,\top\supset B\longrightarrow C}\top\!\supset\!L$$

$$\frac{\Gamma,D\supset E\supset B\longrightarrow C}{\Gamma,D\wedge E\supset B\longrightarrow C}\wedge\!\supset\!L \qquad \frac{\Gamma\longrightarrow C}{\Gamma,\bot\supset B\longrightarrow C}\bot\!\supset\!L \qquad \frac{\Gamma,D\supset B,E\supset B\longrightarrow C}{\Gamma,D\vee E\supset B\longrightarrow C}\vee\!\supset\!L \qquad \frac{\Gamma,D,E\supset B\longrightarrow E \quad \Gamma,B\longrightarrow C}{\Gamma,(D\supset E)\supset B\longrightarrow C}\supset\supset\!L$$

$$\frac{\Gamma\longrightarrow A \quad \Gamma\longrightarrow B}{\Gamma\longrightarrow A\wedge B}\wedge R \qquad \frac{\Gamma,A,B\longrightarrow C}{\Gamma,A\wedge B\longrightarrow C}\wedge L$$

$$\frac{\Gamma\longrightarrow A}{\Gamma\longrightarrow A\vee B}\vee R_1 \qquad \frac{\Gamma\longrightarrow B}{\Gamma\longrightarrow A\vee B}\vee R_2 \qquad \frac{\Gamma,A\longrightarrow C \quad \Gamma,B\longrightarrow C}{\Gamma,A\vee B\longrightarrow C}\vee L$$

$$\frac{}{\Gamma,P\longrightarrow P}\,\mathsf{init} \qquad \frac{}{\Gamma\longrightarrow\top}\top R \qquad \frac{}{\Gamma,\bot\longrightarrow C}\bot L$$

$$\frac{\Gamma\longrightarrow C}{\Gamma,\top\longrightarrow C}\top L$$

[1]The other direction, i.e., the conclusion is justified by the premises, is true for **every** rule.

1.5 Exercises

In the lecture notes it is indicated that cut is admissible for the restricted calculus[2]. The proof is analogous to the one you have already seen, but since less formulas are kept around, some cases become simpler.

Task 1. Prove that if $\Gamma \longrightarrow A \supset B$ and $\Gamma, A \supset B \longrightarrow C$ then $\Gamma \longrightarrow C$ in the restricted sequent calculus (consider only the case where the cut formula is principal).

Solution 1: Assume $\mathcal{D}$ and $\mathcal{E}$ are the following derivations, respectively:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, A \longrightarrow B}}{\Gamma \longrightarrow A \supset B} \supset R \qquad\qquad \dfrac{\dfrac{\mathcal{E}_1}{\Gamma, A \supset B \longrightarrow A} \quad \dfrac{\mathcal{E}_2}{\Gamma \longrightarrow B}}{\Gamma, A \supset B \longrightarrow C} \supset L$$

$\Gamma \longrightarrow A$	by IH on $A \supset B$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma, A \longrightarrow C$	by IH on B, $\mathcal{D}_1$ and $\mathcal{E}_2$
$\Gamma \longrightarrow C$	by IH on A and both previous lines

Task 2. Show that the rules $\wedge \supset L$ and $\vee \supset L$ in G4ip are invertible.

Solution 2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B, D, E \longrightarrow B}\,init}{E \supset B, D, E \longrightarrow B}\,E \supset L}{D \supset E \supset B, D, E \longrightarrow B}\,D \supset L}{D \supset E \supset B \longrightarrow (D \wedge E) \supset B}\,\supset R, \wedge L \qquad \Gamma, (D \wedge E) \supset B \longrightarrow C}{\Gamma, D \supset E \supset B \longrightarrow C}\,cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B, E \supset B, D \longrightarrow B}\,init}{D \supset B, E \supset B, D \longrightarrow B}\,D \supset L \quad \dfrac{\dfrac{}{D \supset B, B, E \longrightarrow B}\,init}{D \supset B, E \supset B, E \longrightarrow B}\,E \supset L}{D \supset B, E \supset B, D \vee E \longrightarrow B}\,\vee L}{D \supset B, E \supset B \longrightarrow D \vee E \supset B}\,\supset R \qquad \Gamma, D \vee E \supset B \longrightarrow C}{\Gamma, D \supset B, E \supset B \longrightarrow C}\,cut$$

Task 3. Prove the following sequent in G4ip:

$$\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R$$

Solution 3:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q \supset R, (P \supset Q) \supset S, Q \longrightarrow Q}\,init}{P, Q \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow Q}\,P \supset L \quad \dfrac{}{R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\,init}{(P \supset Q) \supset R, (P \supset Q) \supset S, (P \supset Q) \longrightarrow R}\,\supset\supset L}{(P \supset Q) \supset R, (P \supset Q) \supset S \longrightarrow (P \supset Q) \supset R}\,\supset R}{((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \longrightarrow (P \supset Q) \supset R}\,\wedge L}{\longrightarrow ((P \supset Q) \supset R) \wedge ((P \supset Q) \supset S) \supset (P \supset Q) \supset R}\,\supset R$$

2 KeYmaera tactics language

You might have noticed that KeYmaera gives you the ability to write which inference rules to apply instead of clicking on the formulas. In fact, it features a *tactic* language, which is simply a language for defining strategies during proof search.

Task 4. Write a tactic in KeYmaera that applies all invertible rules of the sequent calculus eagerly until there is nothing more to do.

Solution 4: `(implyR('R) | andL('L) | andR('R) | andL('L) | orL('L))*`

[2] Actually, cut is admissible for all the calculi listed here.