

1 Concepts

- Why do we use sequent calculus? What are its advantages over natural deduction?
- If I can prove something in sequent calculus can I prove it in natural deduction? What about visa versa?
- What does it mean for sequent calculus to be sound? complete?

Sequent calculus rules

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \quad \frac{\Gamma, B_1 \supset B_2 \Rightarrow B_1 \quad \Gamma, B_1 \supset B_2, B_2 \Rightarrow A}{\Gamma, B_1 \supset B_2 \Rightarrow A} \supset L$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \quad \frac{\Gamma, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_1 \quad \frac{\Gamma, A \wedge B, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L_2$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \vee R_1 \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \vee R_2 \quad \frac{\Gamma, A \vee B, A \Rightarrow C \quad \Gamma, A \vee B, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \vee L$$

$$\overline{\Gamma, P \Rightarrow P} \text{ init} \quad \overline{\Gamma \Rightarrow \top} \top R \quad \overline{\Gamma, \perp \Rightarrow C} \perp L$$

2 Theorems

- **Weakening** For all Γ, A, C , if $\Gamma \Rightarrow C$, then $\Gamma, A \Rightarrow C$.
- **Identity** For all Γ, A : $\Gamma, A \Rightarrow A$.
- **Cut** For all Γ, A, C , if $\Gamma \Rightarrow A$ and $\Gamma, A \Rightarrow C$, then $\Gamma \Rightarrow C$.

3 Sequent Calculus Practice

1. $\Rightarrow A \wedge B \supset B \wedge A$

$$\frac{\frac{\overline{A \wedge B, B \Rightarrow B} \text{ init}}{A \wedge B \Rightarrow B} \wedge L_2 \quad \frac{\overline{A \wedge B, A \Rightarrow A} \text{ init}}{A \wedge B \Rightarrow A} \wedge L_1}{\frac{A \wedge B \Rightarrow B \wedge A}{\Rightarrow A \wedge B \supset B \wedge A} \wedge R} \supset R$$

2. $\Rightarrow A \wedge B \supset A$

$$\frac{\overline{A \wedge B, A \Rightarrow A} \text{ init}}{A \wedge B \Rightarrow A} \wedge L_1 \supset R$$

3. $\Rightarrow A \supset A$

$$\frac{\overline{A \Rightarrow A} \text{ init}}{\Rightarrow A \supset A} \supset R$$

4. $\Rightarrow (B \vee A) \supset (A \vee B)$

$$\frac{\frac{\overline{B \vee A, B \Rightarrow B} \text{ init}}{B \vee A, B \Rightarrow A \vee B} \vee R_2 \quad \frac{\overline{B \vee A, A \Rightarrow A} \text{ init}}{B \vee A, A \Rightarrow A \vee B} \vee R_1}{\frac{(B \vee A) \Rightarrow (A \vee B)}{\Rightarrow (B \vee A) \supset (A \vee B)} \supset R} \vee L$$

5. $\Rightarrow (A \supset (B \wedge C)) \supset (A \supset B)$

$$\frac{\frac{\overline{(A \supset (B \wedge C)), A \Rightarrow A} \text{ init}}{(A \supset (B \wedge C)), A \Rightarrow B} \supset R \quad \frac{\overline{(A \supset (B \wedge C)), A, B \wedge C, B \Rightarrow B} \text{ init}}{(A \supset (B \wedge C)), A, B \wedge C \Rightarrow B} \wedge L_1}{\frac{(A \supset (B \wedge C)), A \Rightarrow B}{(A \supset (B \wedge C)) \Rightarrow (A \supset B)} \supset R} \supset L$$

$$\frac{\frac{(A \supset (B \wedge C)), A \Rightarrow B}{(A \supset (B \wedge C)) \Rightarrow (A \supset B)} \supset R}{\Rightarrow (A \supset (B \wedge C)) \supset (A \supset B)} \supset R$$

6. $\Rightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))$

$$\frac{\frac{\overline{(A \supset B), (A \wedge C), A \Rightarrow A} \text{ init}}{(A \supset B), (A \wedge C) \Rightarrow A} \wedge L_1 \quad \frac{\overline{(A \supset B), (A \wedge C), B \Rightarrow B} \text{ init}}{(A \supset B), (A \wedge C) \Rightarrow B} \supset L \quad \frac{\overline{(A \supset B), (A \wedge C), C \Rightarrow C} \text{ init}}{(A \supset B), (A \wedge C) \Rightarrow C} \wedge L_2}{\frac{(A \supset B), (A \wedge C) \Rightarrow B \wedge C}{(A \supset B) \Rightarrow ((A \wedge C) \supset (B \wedge C))} \supset R} \wedge R$$

$$\frac{\frac{(A \supset B), (A \wedge C) \Rightarrow B \wedge C}{(A \supset B) \Rightarrow ((A \wedge C) \supset (B \wedge C))} \supset R}{\Rightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))} \supset R$$

7. $\Rightarrow ((A \vee B) \wedge C) \supset (A \wedge C) \vee (B \wedge C)$

$$\frac{\frac{\overline{((A \vee B) \wedge C), C, A \vee B, A \Rightarrow A} \text{ init}}{((A \vee B) \wedge C), C, A \vee B, A \Rightarrow (A \wedge C)} \wedge L_1 \quad \frac{\overline{((A \vee B) \wedge C), C, A \vee B, A \Rightarrow C} \text{ init}}{((A \vee B) \wedge C), C, A \vee B, A \Rightarrow (A \wedge C) \vee (B \wedge C)} \wedge R}{\frac{((A \vee B) \wedge C), C, A \vee B, A \Rightarrow (A \wedge C) \vee (B \wedge C)}{((A \vee B) \wedge C), C, A \vee B \Rightarrow (A \wedge C) \vee (B \wedge C)} \vee R_1} \vee L$$

$$\frac{\frac{\overline{((A \vee B) \wedge C), C, A \vee B, B \Rightarrow B} \text{ init}}{((A \vee B) \wedge C), C, A \vee B, B \Rightarrow (B \wedge C)} \wedge L_2 \quad \frac{\overline{((A \vee B) \wedge C), C, A \vee B, B \Rightarrow C} \text{ init}}{((A \vee B) \wedge C), C, A \vee B, B \Rightarrow (A \wedge C) \vee (B \wedge C)} \wedge R}{\frac{((A \vee B) \wedge C), C, A \vee B \Rightarrow (A \wedge C) \vee (B \wedge C)}{((A \vee B) \wedge C), C \Rightarrow (A \wedge C) \vee (B \wedge C)} \wedge L_2} \vee R_2$$

$$\frac{\frac{((A \vee B) \wedge C), C \Rightarrow (A \wedge C) \vee (B \wedge C)}{((A \vee B) \wedge C) \Rightarrow (A \wedge C) \vee (B \wedge C)} \wedge L_1}{\Rightarrow ((A \vee B) \wedge C) \supset (A \wedge C) \vee (B \wedge C)} \supset R$$