Constructive Logic (15-317), Fall 2016
Recitation 2: Tutch and Harmony (9/6/2016)
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Part I
Tutch

Prove the following theorems using tutch (if you find it easier, construct the proofs by hand first).

```
proof curry : (A & B => C) => (A => B => C) =
begin
[ A & B => C;
  [ A;
    [ B;
      A & B;
      C ];
    B => C ];
  A => B => C;
];
(A & B => C) => (A => B => C);
end;
```

```
proof uncurry : (A => B => C) => (A & B => C) =
begin
[ A => B => C;
  [ A & B;
    A;
    B => C;
    B;
    C ];
  A & B => C;
];
(A => B => C) => (A & B => C);
end;
```

```
proof distributivity_expand : A | (B & C) => (A | B) & (A | C) =
begin
[ A | (B & C);
    [ A;
      A | B ];
    [ B & C;
      B;
      A | B ];
  A | B;
  A | (B & C);
    [ A;
      A | C ];
    [ B & C;
      C;
      A | C ];
  A | C;
  (A | B) & (A | C);
];
A | (B & C) => (A | B) & (A | C);
end;
```

Part II
Harmony

Today we'll look at some examples using harmony to show that a new logical connective is or is not well-designed.

Let's define two ternary logical connectives, $\heartsuit$ and $\diamond$, and see if they are locally sound and complete.

$$\frac{\begin{array}{cc} \overline{A\ true}^{\,u} & \overline{B\ true}^{\,v} \\ \vdots & \vdots \\ C\ true & C\ true \end{array}}{\heartsuit(A,B,C)\ true}\ \heartsuit I^{u,v} \qquad \frac{\heartsuit(A,B,C)\ true \quad A\ true}{C\ true}\ \heartsuit E_1 \qquad \frac{\heartsuit(A,B,C)\ true \quad B\ true}{C\ true}\ \heartsuit E_2$$

Solution:
Local soundness:

$$\frac{\dfrac{\begin{array}{cc} \overline{A\ true}^{\,u} & \overline{B\ true}^{\,v} \\ \mathcal{D} & \mathcal{E} \\ C\ true & C\ true \end{array}}{\heartsuit(A,B,C)\ true}\ \heartsuit I^{u,v} \quad \begin{array}{c} \mathcal{F} \\ A\ true \end{array}}{C\ true}\ \heartsuit E_1 \quad \Longrightarrow_R \quad \begin{array}{c} [\mathcal{F}/u]\mathcal{D} \\ C\ true \end{array}$$

$$\frac{\dfrac{\begin{array}{cc} \overline{A\ true}^{\,u} & \overline{B\ true}^{\,v} \\ \mathcal{D} & \mathcal{E} \\ C\ true & C\ true \end{array}}{\heartsuit(A,B,C)\ true}\ \heartsuit I^{u,v} \quad \begin{array}{c} \mathcal{G} \\ B\ true \end{array}}{C\ true}\ \heartsuit E_2 \quad \Longrightarrow_R \quad \begin{array}{c} [\mathcal{G}/v]\mathcal{E} \\ C\ true \end{array}$$

Local completeness:

$$\begin{array}{c} \mathcal{D} \\ \heartsuit(A,B,C)\ true \end{array} \quad \Longrightarrow_E \quad \frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \heartsuit(A,B,C)\ true\end{array} \quad \overline{A\ true}^{\,u}}{C\ true} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \heartsuit(A,B,C)\ true\end{array} \quad \overline{B\ true}^{\,v}}{C\ true}}{\heartsuit(A,B,C)\ true}\ \heartsuit I^{u,v}$$

$$\frac{\begin{array}{c} \overline{A\ true}^{\,u} \\ \vdots \\ C\ true \end{array}}{\diamond(A,B,C)\ true}\ \diamond I_1^u \qquad \frac{\begin{array}{c} \overline{B\ true}^{\,v} \\ \vdots \\ C\ true \end{array}}{\diamond(A,B,C)\ true}\ \diamond I_2^v \qquad \frac{\diamond(A,B,C)\ true \quad A\ true \quad B\ true}{C\ true}\ \diamond E$$

Solution:
Local soundness:

$$\frac{\dfrac{\begin{array}{c} \overline{A\ true}^{\,u} \\ \mathcal{F} \\ C\ true \end{array}}{\diamond(A,B,C)\ true}\ \diamond I_1^u \quad \begin{array}{c} \mathcal{D} \\ A\ true \end{array} \quad \begin{array}{c} \mathcal{E} \\ B\ true \end{array}}{C\ true}\ \diamond E \quad \Longrightarrow_R \quad \begin{array}{c} [\mathcal{D}/u]\mathcal{F} \\ C\ true \end{array}$$

$$\frac{\dfrac{\begin{array}{c} \overline{B\ true}^{\,v} \\ \mathcal{G} \\ C\ true \end{array}}{\diamond(A,B,C)\ true}\ \diamond I_2^v \quad \begin{array}{c} \mathcal{D} \\ A\ true \end{array} \quad \begin{array}{c} \mathcal{E} \\ B\ true \end{array}}{C\ true}\ \diamond E \quad \Longrightarrow_R \quad \begin{array}{c} [\mathcal{E}/v]\mathcal{G} \\ C\ true \end{array}$$

Local completeness does not hold. The final rule has to introduce $\diamond(A,B,C)$, so it must be either $\diamond I_1$ or $\diamond I_2$. Then you have an assumption either of $A\ true$ or of $B\ true$, but neither of these alone is enough to apply $\diamond E$.

The failure of completeness corresponds to the fact that $(A \supset C) \vee (B \supset C)$ (which is characterized by the $\diamond$ introduction rules) is stronger than $(A \wedge B) \supset C$ (which is characterized by the $\diamond$ elimination rules). In classical logic, these are equivalent, but this is not the case constructively!