Midterm I Exam

15-317/657 Constructive Logic
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September 29, 2016

Name: ______________________________

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Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 6 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
New Connections	50	
Harmonic Series	20	
Using Verifications	40	
Recurse Primitively	40	
Total:	150	

1 New Connections (50 points)

Consider the new connective $\square(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$\dfrac{\begin{array}{cc} \dfrac{}{B\ \mathit{true}}u & \dfrac{}{C\ \mathit{true}}w \\ \vdots & \vdots \\ A\ \mathit{true} & A\ \mathit{true} \end{array}}{\square(A, B, C)\ \mathit{true}}\ \square I^{u,w}$$

$\boxed{10}$ **Task 1** Give the elimination rule(s) that harmoniously fit to $\square I$:

$\boxed{10}$ **Task 2** Prove local soundness for the $\square$ connective.

$\boxed{10}$ **Task 3** Prove local completeness for the $\Box$ connective.

$\boxed{10}$ **Task 4** Propose a proof term assignment for all rules of the $\Box$ connective.

$\boxed{10}$ **Task 5** Provide local reduction rules for the proof terms of the $\Box$ connective.

2 Harmonic Series (20 points)

Detective Chase McCase is hunting a group of disharmonious crooks, who wrote down random introduction and elimination rules. Help him sort it out by marking connectives as:

ⓗ for harmonious connectives and provide a local reduction on proofs or proof terms.

ⓤ for unharmonious connectives and explain **in one sentence** one case that fails and why.

Note: for harmonious connectives, you do not need to write down the local completeness argument (but make sure it is locally sound and locally complete).

$\boxed{10}$ **Task 1**

$$\dfrac{\begin{array}{c}\overline{u : A}\ u \\ \vdots \\ M : B\end{array}}{m(u{:}A, M) : A \succ\!\!- B} \succ\!\!-I^u \qquad \dfrac{M : A \succ\!\!- B}{r(M) : B} \succ\!\!-E$$

$\boxed{10}$ **Task 2**

$$\dfrac{M : A \quad N : B}{s(M, N) : A \Diamond B} \Diamond I \qquad \dfrac{M : A \Diamond B}{f(M) : A} \Diamond E$$

3 Using Verifications (40 points)

The lectures studied natural deduction with proof rules for the truth judgment of the form A *true* as well as verification rules for verifications of the form $A\uparrow$.

$\boxed{20}$ **Task 1** Give a proof of $A \supset A$ in natural deduction that is not also a verification (meaning replacing all C *true* by one of $C\uparrow$ or $C\downarrow$) and briefly indicate why it is not a verification. Or briefly explain why that is impossible.

$\boxed{20}$ **Task 2** Give a verification of $A \supset A$ that is not also a natural deduction proof (meaning replacing all $C\uparrow$ or $C\downarrow$ by C *true*) and briefly indicate why it is not a proof of truth in natural deduction. Or briefly explain why that is impossible.

4 Recurse Primitively (40 points)

Your homework showed that multiplication $\mathsf{mult} : \mathsf{nat} \to (\mathsf{nat} \to \mathsf{nat})$ on natural numbers is primitively recursive:

$$\begin{array}{lcl} \mathsf{mult}(0) & = & \lambda y.\, 0 \\ \mathsf{mult}(\mathsf{s}\, n) & = & \lambda y.\, \Big(\mathsf{plus}\, \big((\mathsf{mult}\, n)\, y\big)\Big)\, y \end{array}$$

Are the following function definitions primitively recursive? If so, give an equivalent proof term using $R(n,\, t_0,\, x.r.t_s)$. Otherwise explain why the definition is not primitively recursive.

$\boxed{20}$ **Task 1**

$$\mathsf{p}(0) = \lambda y.\, \mathsf{s}\, 0$$
$$\mathsf{p}(\mathsf{s}\, n) = \lambda y.\, \Big(\mathsf{mult}\big((\mathsf{p}\, n)\, y\big)\Big)\, y$$

$\boxed{20}$ **Task 2**

$$\mathsf{q}(0) = 0$$
$$\mathsf{q}(\mathsf{s}\, n) = \Big(\mathsf{mult}\big(\mathsf{s}\, \mathsf{s}(\mathsf{q}\, n)\big)\Big)\, \big(\mathsf{s}(\mathsf{q}\, n)\big)$$