Midterm I Exam

15-317/657 Constructive Logic
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Name: 
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Instructions

• This exam is closed-book with one sheet of notes permitted.
• You have 80 minutes to complete the exam.
• There are 4 problems on 6 pages.
• Read each problem carefully before attempting to solve it.
• Do not spend too much time on any one problem.
• Consider if you might want to skip a problem on a first pass and return to it later.

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1 New Connections (50 points)
Consider the new connective $\Box(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$
\begin{array}{c}
B \text{ true} \\
\vdots \\
A \text{ true} \\
\hline
\Box(A, B, C) \text{ true}
\end{array}
\Downarrow^{w}
\begin{array}{c}
C \text{ true} \\
\vdots \\
A \text{ true} \\
\hline
\Box(A, B, C) \text{ true}
\end{array}
\Downarrow^{u,w}

10 Task 1 Give the elimination rule(s) that harmoniously fit to $\Box I$:

10 Task 2 Prove local soundness for the $\Box$ connective.
Task 3  Prove local completeness for the □ connective.

Task 4  Propose a proof term assignment for all rules of the □ connective.

Task 5  Provide local reduction rules for the proof terms of the □ connective.
2 Harmonic Series (20 points)

Detective Chase McCase is hunting a group of disharmonious crooks, who wrote down random introduction and elimination rules. Help him sort it out by marking connectives as:

③ for harmonious connectives and provide a local reduction on proofs or proof terms.
④ for unharmonious connectives and explain in one sentence one case that fails and why.

Note: for harmonious connectives, you do not need to write down the local completeness argument (but make sure it is locally sound and locally complete).

10 Task 1

\[
\frac{u : A}{\vdots} \quad \frac{M : B}{m(u:A,M) : A \rightarrow B \quad M : A \rightarrow B \quad r(M) : B \quad \rightarrow E}
\]

10 Task 2

\[
\frac{M : A \quad N : B}{s(M,N) : A \diamond B \quad \diamond I} \quad \frac{M : A \diamond B}{f(M) : A \quad \diamond E}
\]
3 Using Verifications (40 points)

The lectures studied natural deduction with proof rules for the truth judgment of the form $A \text{ true}$ as well as verification rules for verifications of the form $A \uparrow$.

**Task 1** Give a proof of $A \supset A$ in natural deduction that is not also a verification (meaning replacing all $C \text{ true}$ by one of $C \uparrow$ or $C \downarrow$) and briefly indicate why it is not a verification. Or briefly explain why that is impossible.

**Task 2** Give a verification of $A \supset A$ that is not also a natural deduction proof (meaning replacing all $C \uparrow$ or $C \downarrow$ by $C \text{ true}$) and briefly indicate why it is not a proof of truth in natural deduction. Or briefly explain why that is impossible.
4 Recurse Primitively (40 points)
Your homework showed that multiplication $\text{mult} : \text{nat} \to (\text{nat} \to \text{nat})$ on natural numbers is primitively recursive:

$$\begin{align*}
\text{mult}(0) &= \lambda y. 0 \\
\text{mult}(s\ n) &= \lambda y. \left(\text{plus}\ ((\text{mult}\ n)\ y)\ y\right)
\end{align*}$$

Are the following function definitions primitively recursive? If so, give an equivalent proof term using $R(n, t_0, x.r.t_s)$. Otherwise explain why the definition is not primitively recursive.

**Task 1**

\[p(0) = \lambda y. s\ 0\]
\[p(s\ n) = \lambda y. \left(\text{mult}\ ((p\ n)\ y)\ y\right)\]

**Task 2**

\[q(0) = 0\]
\[q(s\ n) = \left(\text{mult}\ (s\ s(q\ n))\right)\ (s(q\ n))\]