Midterm I Exam

15-317/657 Constructive Logic
André Platzer

September 29, 2016

Name: André Platzer

Andrew ID: aplatzer

Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 80 minutes to complete the exam.
- There are 4 problems on 6 pages.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

	Max	Score
New Connections	50	
Harmonic Series	20	
Using Verifications	40	
Recurse Primitively	40	
Total:	150	

Please keep in mind that this is a sample solution*, not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.*

1 New Connections (50 points)

Consider the new connective $\square(A, B, C)$ that your friendly verificationists gave meaning to by the following introduction rule:

$$\frac{\begin{array}{cc} \overline{B\ \mathit{true}}^{\,u} & \overline{C\ \mathit{true}}^{\,w} \\ \vdots & \vdots \\ A\ \mathit{true} & A\ \mathit{true} \end{array}}{\square(A, B, C)\ \mathit{true}}\ \square I^{u,w}$$

[10] **Task 1** Give the elimination rule(s) that harmoniously fit to $\square I$:

Solution:

$$\frac{\square(A, B, C)\ \mathit{true} \quad B\ \mathit{true}}{A\ \mathit{true}}\ \square E_1 \qquad \frac{\square(A, B, C)\ \mathit{true} \quad C\ \mathit{true}}{A\ \mathit{true}}\ \square E_2$$

[10] **Task 2** Prove local soundness for the $\square$ connective.

Solution:

$$\frac{\dfrac{\begin{array}{cc} \overline{B\ \mathit{true}}^{\,u} & \overline{C\ \mathit{true}}^{\,w} \\ \mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & A\ \mathit{true} \end{array}}{\square(A, B, C)\ \mathit{true}}\ \square I^{u,w} \qquad \begin{array}{c} \mathcal{F} \\ B\ \mathit{true} \end{array}}{A\ \mathit{true}}\ \square E_1 \quad \Rightarrow_R \quad \begin{array}{c} \mathcal{F} \\ \overline{B\ \mathit{true}}^{\,u} \\ \mathcal{D} \\ A\ \mathit{true} \end{array}$$

$$\frac{\dfrac{\begin{array}{cc} \overline{B\ \mathit{true}}^{\,u} & \overline{C\ \mathit{true}}^{\,w} \\ \mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & A\ \mathit{true} \end{array}}{\square(A, B, C)\ \mathit{true}}\ \square I^{u,w} \qquad \begin{array}{c} \mathcal{F} \\ C\ \mathit{true} \end{array}}{A\ \mathit{true}}\ \square E_2 \quad \Rightarrow_R \quad \begin{array}{c} \mathcal{F} \\ \overline{C\ \mathit{true}}^{\,u} \\ \mathcal{E} \\ A\ \mathit{true} \end{array}$$

10 **Task 3** Prove local completeness for the $\Box$ connective.

Solution:

$$\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \Rightarrow_E \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \quad \dfrac{}{B\ \mathit{true}}u}{A\ \mathit{true}}\Box E_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Box(A,B,C)\ \mathit{true}\end{array} \quad \dfrac{}{C\ \mathit{true}}w}{A\ \mathit{true}}\Box E_2}{\Box(A,B,C)\ \mathit{true}}\Box I^{u,w}$$

10 **Task 4** Propose a proof term assignment for all rules of the $\Box$ connective.

Solution:

$$\dfrac{\begin{array}{c}\dfrac{}{u : B}u\\ \vdots\\ M : A\end{array} \quad \begin{array}{c}\dfrac{}{w : C}w\\ \vdots\\ N : A\end{array}}{b(u.M, w.N) : \Box(A,B,C)}\Box I^{u,w}$$

$$\dfrac{M : \Box(A,B,C) \quad N : B}{l(M,N) : A}\Box E_1$$

$$\dfrac{M : \Box(A,B,C) \quad N : C}{r(M,N) : A}\Box E_2$$

10 **Task 5** Provide local reduction rules for the proof terms of the $\Box$ connective.

Solution:

$$l(b(u.M, w.N), L) \Rightarrow_R [L/u]M$$
$$r(b(u.M, w.N), R) \Rightarrow_R [R/w]N$$

2 Harmonic Series (20 points)

Detective Chase McCase is hunting a group of disharmonious crooks, who wrote down random introduction and elimination rules. Help him sort it out by marking connectives as:

- ⓗ for harmonious connectives and provide a local reduction on proofs or proof terms.
- ⓤ for unharmonious connectives and explain **in one sentence** one case that fails and why.

Note: for harmonious connectives, you do not need to write down the local completeness argument (but make sure it is locally sound and locally complete).

10 **Task 1**

$$\dfrac{\begin{array}{c}\overline{u : A}\ u\\ \vdots \\ M : B\end{array}}{m(u{:}A, M) : A \succ\!\!- B} \succ\!\!-I^u \qquad \dfrac{M : A \succ\!\!- B}{r(M) : B} \succ\!\!-E$$

Solution: ⓤ unsound since B has no unconditional proof, only a proof with an unjustified additional assumption $u : A$

10 **Task 2**

$$\dfrac{M : A \quad N : B}{s(M, N) : A \Diamond B} \Diamond I \qquad \dfrac{M : A \Diamond B}{f(M) : A} \Diamond E$$

Solution: ⓤ incomplete since there is no elimination rule to get B back out of $s(M, N) : A \Diamond B$.

3 Using Verifications (40 points)

The lectures studied natural deduction with proof rules for the truth judgment of the form A *true* as well as verification rules for verifications of the form $A\uparrow$.

20 **Task 1** Give a proof of $A \supset A$ in natural deduction that is not also a verification (meaning replacing all C *true* by one of $C\uparrow$ or $C\downarrow$) and briefly indicate why it is not a verification. Or briefly explain why that is impossible.

Solution: Proof of $A \supset A$ *true* using an extra irrelevant redundant proof of $A \supset A$ *true*.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,w}{A \supset A\ \mathit{true}}\,{\supset}I^w \quad \dfrac{}{A\ \mathit{true}}\,u}{A\ \mathit{true}}\,{\supset}E}{A \supset A\ \mathit{true}}\,{\supset}I^u$$

This proof does not correspond to a verification, because there is no license ot use $A \supset A\downarrow$ at the indicated step.

20 **Task 2** Give a verification of $A \supset A$ that is not also a natural deduction proof (meaning replacing all $C\uparrow$ or $C\downarrow$ by C *true*) and briefly indicate why it is not a proof of truth in natural deduction. Or briefly explain why that is impossible.

Solution: That is impossible since verification implies truth, since all verification and uses rule have the same shape as the corresponding rule for truth. So eliding $\uparrow$ and $\downarrow$ and replacing them with the judgment *true* leads to a proof after skipping the $\uparrow\downarrow$ rule for atomic propositions.

4 Recurse Primitively (40 points)

Your homework showed that multiplication $\mathsf{mult} : \mathsf{nat} \to (\mathsf{nat} \to \mathsf{nat})$ on natural numbers is primitively recursive:

$$\begin{aligned} \mathsf{mult}(0) &= \lambda y.\, 0 \\ \mathsf{mult}(\mathsf{s}\, n) &= \lambda y.\, \Big(\mathsf{plus}\ \big((\mathsf{mult}\, n)\, y\big)\Big)\ y \end{aligned}$$

Are the following function definitions primitively recursive? If so, give an equivalent proof term using $R(n,\, t_0,\, x.r.t_s)$. Otherwise explain why the definition is not primitively recursive.

20 **Task 1**

$$\mathsf{p}(0) = \lambda y.\, \mathsf{s}\, 0$$
$$\mathsf{p}(\mathsf{s}\, n) = \lambda y.\, \Big(\mathsf{mult}\big((\mathsf{p}\, n)\, y\big)\Big)\ y$$

Solution: yes

$$p = \lambda n.\ R\Big(n,\ \lambda y.\, \mathsf{s}0,\ x.r.\lambda y.\, \big(\mathsf{mult}(r\, y)\big)\ y\Big)$$

20 **Task 2**

$$\mathsf{q}(0) = 0$$
$$\mathsf{q}(\mathsf{s}\, n) = \Big(\mathsf{mult}\big(\mathsf{s}\, \mathsf{s}(\mathsf{q}\, n)\big)\Big)\ \big(\mathsf{s}(\mathsf{q}\, n)\big)$$

Solution: yes even if it mentions the recursion twice it's still the same recursive value r

$$q = \lambda n.\ R\Big(n,\ 0,\ x.r.\big(\mathsf{mult}(\mathsf{s}\, \mathsf{s}\, r)\big)\ \big(\mathsf{s}\, r\big)\Big)$$