

natural deduction

$$\frac{\frac{A \text{ true} \quad B \text{ true}}{A \supset B \text{ true}} \supset I^u \quad \varepsilon}{A \supset B \text{ true} \quad A \text{ true} \supset E} \Rightarrow R$$

$B \text{ true}$

$$\frac{\frac{A \supset B \text{ true} \quad A \text{ true}}{B \text{ true}} \supset E \quad \varepsilon}{A \text{ true} \quad B \text{ true}} \wedge I^u$$

Constructive truth := proof (PML verified)
 classical truth :=

classical

		B	
A ⊃ B		Y	N
Y	N	Y	N
Y	Y	Y	Y

A	¬A
Y	N
N	Y

A ∨ B	Y	N
Y	N	Y
N	Y	N

$A \supset B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$
 $A \vee B \equiv (A \vee B) \wedge (A \wedge B)$

REVIEW SESSION

natural deduction

$$\frac{\overline{A \text{ true}}^u}{B \text{ true}} \Rightarrow \text{I}^u \quad \frac{\overline{A \supset B \text{ true}}}{B \text{ true}} \Rightarrow \text{E}$$

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$$\frac{\overline{A \supset B \text{ true}}}{B \text{ true}} \Rightarrow \text{I}^u \quad \frac{\overline{A \text{ true}}}{B \text{ true}} \Rightarrow \text{E} \Rightarrow \text{loc. sound} \Rightarrow \text{R}$$

$$\frac{\overline{A \supset B \text{ true}}}{B \text{ true}} \Rightarrow \text{I}^u \quad \frac{\overline{A \text{ true}}}{B \text{ true}} \Rightarrow \text{E} \Rightarrow \text{loc. sound} \Rightarrow \text{R} \Rightarrow \text{loc. complete}$$

$$\frac{\overline{A \supset B \text{ true}}}{B \text{ true}} \Rightarrow \text{I}^u \quad \frac{\overline{A \text{ true}}}{B \text{ true}} \Rightarrow \text{E} \Rightarrow \text{loc. sound} \Rightarrow \text{R} \Rightarrow \text{loc. complete}$$

verifications & uses

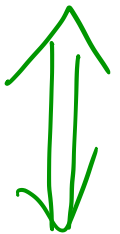
$$\frac{\overline{A \downarrow}^u}{B \uparrow} \Rightarrow \text{I}^u \quad \frac{\overline{A \supset B \downarrow}}{B \downarrow} \Rightarrow \text{E}$$

$$\frac{\overline{A \downarrow}^u}{B \uparrow} \Rightarrow \text{I}^u \quad \frac{\overline{A \supset B \downarrow}}{B \downarrow} \Rightarrow \text{E}$$

$$\frac{\overline{A \downarrow}^u}{B \uparrow} \Rightarrow \text{I}^u \quad \frac{\overline{A \supset B \downarrow}}{B \downarrow} \Rightarrow \text{E} \Rightarrow \text{loc. sound} \Rightarrow \text{R}$$

$$\frac{\overline{A \downarrow}^u}{B \uparrow} \Rightarrow \text{I}^u \quad \frac{\overline{A \supset B \downarrow}}{B \downarrow} \Rightarrow \text{E} \Rightarrow \text{loc. sound} \Rightarrow \text{R} \Rightarrow \text{loc. complete}$$

cut: γ



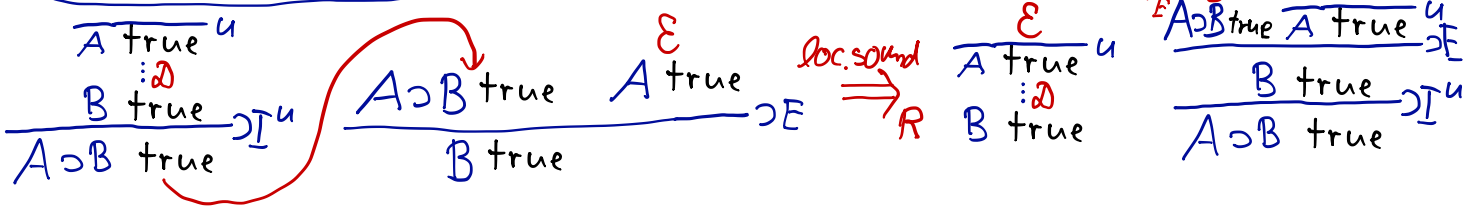
$A \uparrow$ and $A \downarrow$ then $B \uparrow$
 $\text{cut } \gamma \quad \Gamma \Rightarrow A \text{ and } \Gamma, A \Rightarrow B \text{ then } \Gamma \Rightarrow B$
 identity $\frac{A \downarrow}{A \uparrow} \text{ id} \quad A \Rightarrow A$

sequent calculus

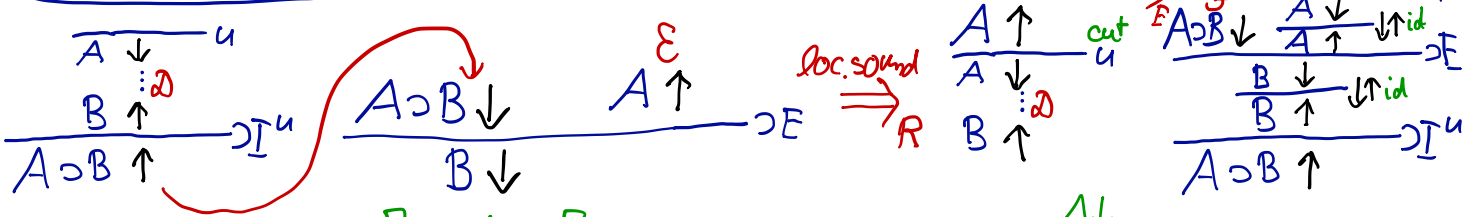
$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \Rightarrow \text{R}$$

$$\frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \Rightarrow \text{L}$$

natural deduction



verifications & uses

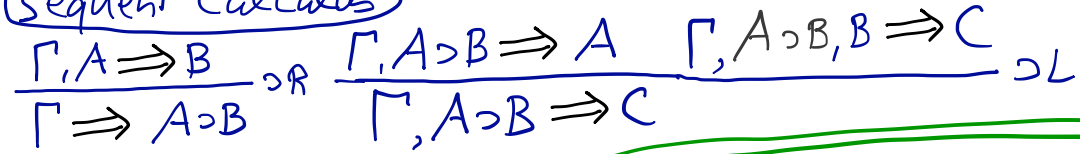


Thm(Cut): $\frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C} \text{cut}$ *glob. sound* $A \uparrow$ $\frac{A \downarrow}{C \uparrow}$ *than* $C \uparrow$

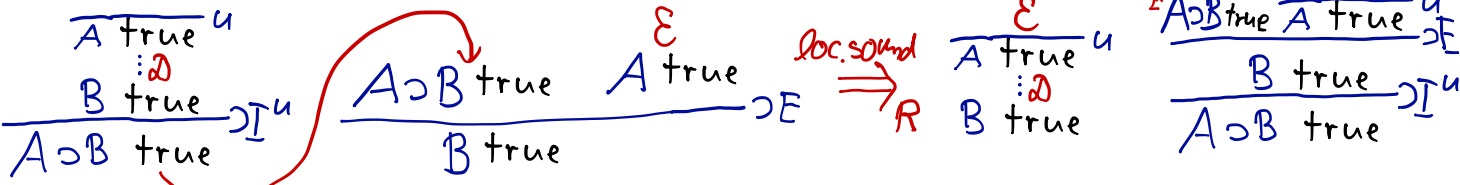
Thm(Id): $A \Rightarrow A \text{ id}$

glob. complete $\frac{A \downarrow}{A \uparrow}$

sequent calculus

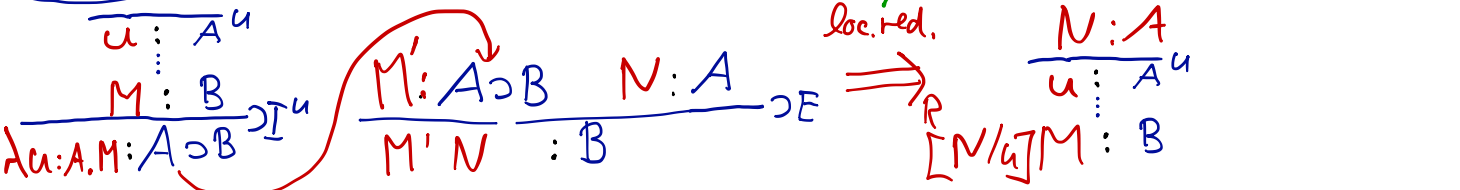


natural deduction



proof terms

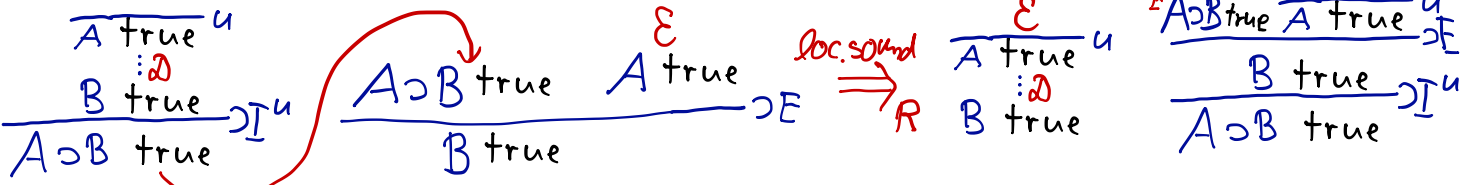
propositions as types proofs as programs



$$\begin{aligned}
 (\lambda u:A. M) N &\Rightarrow_R [N/u] M \\
 M : A > B &\Rightarrow_E (\lambda u:A. M u) : A > B
 \end{aligned}$$

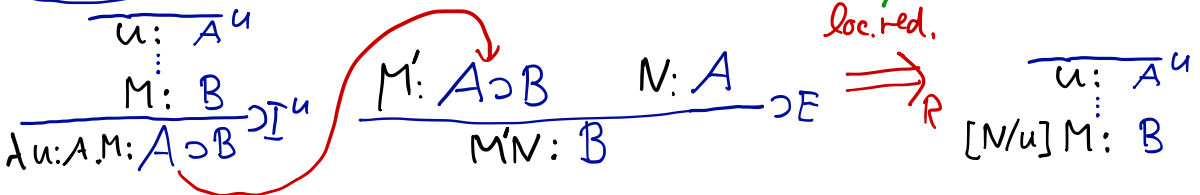
logic programming

natural deduction



proof terms

propositions as types proofs as programs



$$(\lambda u:A.M)N \Rightarrow_R [N/u]M$$

$$M:A \Rightarrow_F \lambda u:A. Mu:A$$

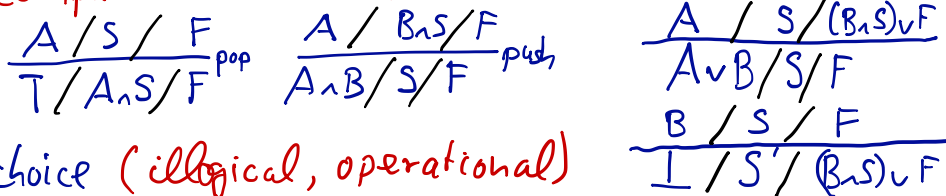
$u \notin FV(M)$

logic programming

formulas as programs
Computational interpretation of proof search

Fix strategy

- 1 left-to-right subgoals
- 2 First-to-last clause
- 3 Unification algo
- 4 Cut commits to clause choice (illogical, operational)



sequent calculus

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \quad \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L$$

restrictive sequent calculus

sequent calculus

direct link to natural deduction

$\Gamma \vdash C$
 \Downarrow
 $\Gamma \Rightarrow C$
 \Downarrow

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \quad \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L$$

restrictive sequent calculus

no redundancy except $\supset L$

$$\frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B} \supset R \quad \frac{\Gamma, A \supset B \rightarrow A \quad \Gamma, B \rightarrow C}{\Gamma, A \supset B \rightarrow C} \supset L$$

$$\frac{\Gamma, A, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge R \quad \Gamma \rightarrow C \quad (\Downarrow)$$

invertible sequent calculus

no choice except $\supset L$ and $\vee R$

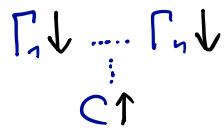
$$\frac{\Gamma^-; \Omega, A, B \rightarrow^+ C^+}{\Gamma^-; \Omega, A \wedge B \rightarrow^+ C^+} \wedge L \quad \frac{\Gamma^-; \Omega, A \rightarrow^+ C^+ \quad \Gamma^-; \Omega, B \rightarrow^+ C^+}{\Gamma^-; \Omega, A \vee B \rightarrow^+ C^+} \vee L$$

invertible so forced
 (\Downarrow)

Contraction-free calculus

no search except $\supset L$ and $\vee R$

$$\frac{\Gamma, D \supset B, E \supset B \rightarrow C}{\Gamma, D \vee E \supset B \rightarrow C} \vee \supset L \quad \frac{\Gamma, D \supset (E \supset B) \rightarrow C}{\Gamma, D \wedge E \supset B \rightarrow C} \wedge \supset L \quad \frac{\Gamma, E \supset B, D \rightarrow E \quad \Gamma, B \rightarrow C}{\Gamma, (D \supset E) \supset B \rightarrow C} \supset \supset L$$



\Downarrow

cut-free

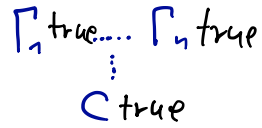
$\Gamma_1, \dots, \Gamma_n \Rightarrow C$

\Downarrow

with cut

$\Gamma_1, \dots, \Gamma_n \Rightarrow C$

\Downarrow



$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I$$

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_1$$



$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R$$

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \wedge L$$