Linear Logic

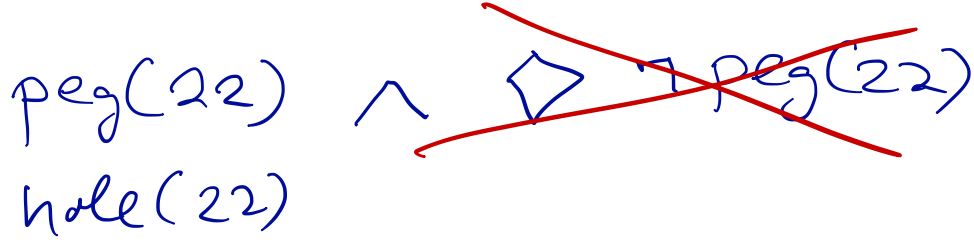
imperative programming
logic of state change
treat state logically, not as explicit data structure

State-Passing Logic Programming

Recall

```
activate(A, OldState, NewState) :- .....
deactivate(A, OldState, NewState) :- .....

transformation(InputState, OutputState) :-
    activate(a, InputState, Intermediate),
    deactivate(b, Intermediate, OutputState).
```



Chemical Linear Logic Example

~~water ⊸ gold~~

⊸ Lolli

⊗ simultaneous conjunction

$2H_2 + O_2 \rightarrow H_2O$

$!(H_2 \otimes H_2 \otimes O_2 \multimap water \otimes water)$

H_2O_2

$H_2 \otimes O_2 \multimap$ Hydrogen Peroxide

& alternative conjunction

$H_2 \otimes H_2 \otimes O_2 \multimap (water \otimes water) \,\&\, (peroxide \otimes H_2)$

$H_2 \otimes H_2 \otimes O_2 \multimap (water \otimes water) \oplus (peroxide \otimes H_2)$

⊕ disjunction

$Peroxide \otimes Peroxide \otimes Pt \multimap water \otimes water \otimes O_2 \otimes Pt$

catalyst — Pt enables reaction not consumed

$Pt \multimap 1$

1 empty truth

$!A$

$Energy \multimap 1$

Linear Logic

$A_1\ res, \ldots, A_n\ res \vdash C\ true$

Ai linear assumptions are used exactly once

ephemeral truth in current state, not permanent truth

Connectives of (Intuitionistic) Linear Logic

$$A ::= A \otimes B \mid A \,\&\, B \mid A \multimap B \mid 1 \mid \top \mid 0 \mid A \oplus B \mid\ !A$$

$$\frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B}\ \otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L$$

$$\frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\ \&R \qquad \frac{\Delta, A \vdash C}{\Delta, A \,\&\, B \vdash C}\ \&L_1 \qquad \frac{\Delta, B \vdash C}{\Delta, A \,\&\, B \vdash C}$$

$$\frac{}{\Delta \vdash \top}\ \top R \qquad \text{no } \top L$$

$$\frac{}{\cdot \vdash 1}\ 1R \qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C}\ 1L$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \multimap R \qquad \frac{\Delta \vdash A \quad \Delta', B \vdash C}{\Delta, \Delta', A \multimap B \vdash C}\ \multimap L$$

$$\frac{}{P \vdash P}\ init$$

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L \qquad \frac{\Gamma, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\,\oplus R_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\,\oplus R_2$$

$$\frac{\Delta, A \vdash C \qquad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C}\,\oplus L$$

$$\frac{\Gamma;\ \Delta \vdash A \qquad \Gamma;\ \Delta \vdash B}{\Gamma;\ \Delta \vdash A \,\&\, B}\,\&R \qquad \frac{\Gamma;\ \Delta, A \vdash C}{\Gamma;\ \Delta, A \,\&\, B \vdash C}\,\&L_1 \qquad \frac{\Gamma;\ \Delta, B \vdash C}{\Gamma;\ \Delta, A \,\&\, B \vdash C}$$

$$\frac{\Gamma;\ \cdot \vdash A}{\Gamma;\ \cdot \vdash\ !A}\,!R \qquad \frac{\Gamma, A;\ \Delta \vdash C}{\Gamma;\ \Delta, !A \vdash C}\,!L$$

$$\frac{}{\Gamma;\ \Delta, 0 \vdash C}\,0L \qquad \frac{\Gamma, A;\ \Delta, A \vdash C}{\Gamma, A;\ \Delta \vdash C}\,\text{copy}$$

$$\frac{\Delta, !A, A \vdash C}{\Delta, !A \vdash C}\,!L \qquad \frac{\Delta \vdash C}{\Delta, !A \vdash C}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{init} \qquad \dfrac{}{B \vdash B}\,\text{init}}{A \multimap B,\ A \vdash B}\,\multimap L}{A \multimap B \vdash A \multimap B}\,\multimap R}{\vdash (A \multimap B) \multimap (A \multimap B)}\,\multimap R$$

Rewrite resource gone

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \multimap B;\ A \vdash A}\,\text{init} \qquad \dfrac{}{A \multimap B;\ B \vdash B}\,\text{init}}{A \multimap B;\ A,\ A \multimap B \vdash B}}{A \multimap B;\ A \vdash B}\,\text{copy}}{A \multimap B;\ \vdash A \multimap B}\,\multimap R}{;\ !(A \multimap B) \vdash A \multimap B}\,!L}{;\ \vdash\ !(A \multimap B) \multimap (A \multimap B)}\,\multimap R$$

reusable rewrite resource $!(A \multimap B)$

Linear Logic

$$\frac{}{P \vdash P}\ id \qquad \frac{}{\Delta, 0 \vdash C}\ 0L$$

$$\frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B}\otimes R \qquad \frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\otimes L$$

$$\frac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\,\&R \qquad \frac{\Delta, A \vdash C}{\Delta, A \,\&\, B \vdash C}\,\&L_1 \qquad \frac{\Delta, B \vdash C}{\Delta, A \,\&\, B \vdash C}\,\&L_2$$

$$\frac{}{\Delta \vdash \top}\top R \qquad \text{no } \top L$$

$$\frac{}{\cdot \vdash 1}\,1R \qquad \frac{\Delta \vdash C}{\Delta, 1 \vdash C}\,1L$$

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\multimap R \qquad \frac{\Delta \vdash A \quad \Delta', B \vdash C}{\Delta, \Delta', A \multimap B \vdash C}\multimap L$$

$$\frac{\Gamma; \cdot \vdash A}{\Gamma; \cdot \vdash\, !A}\,!R \qquad \frac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C}\,!L \qquad \frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}\ copy$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\oplus R_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\oplus R_2 \qquad \frac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C}\oplus L$$