Resolution

Robinson '65

- Resolution with unification as the central component for automated theorem proving in classical FOL
- Practical procedure still today (with many optimizations)
- Earlier procedures have now become practical too '20 '35 ...

Classical First-Order Logic

Frege 1879

Syntax $G, F ::= p(\bar{t}) \mid \neg F \mid F \wedge G \mid F \vee G \mid F \to G \mid \forall x F \mid \exists x F$

Semantics Interpretation I fixes a set U as universe

$I(p) \subseteq U^n = U \times \dots \times U$ predicate symbol p/n

$I(f) : U^n \to U$ function symbol f/n

$\eta : Var \to U$ assigns

$[\![p(\bar{t})]\!] = \{ \eta \mid (\text{value of } \bar{t} \text{ in } \eta) \in I(p) \}$

$\exists x\, p(x,y) \wedge \neg \forall x\, p(x,y)$

$U = \{R, S, C\}$

$I(p) = \{(R,C), (S,C)\}$

$(\eta(x), \eta(y)) \in I(p)$

$[\![F \wedge G]\!] = [\![F]\!] \cap [\![G]\!]$

$[\![F \vee G]\!] = [\![F]\!] \cup [\![G]\!]$

$[\![\neg F]\!] = [\![F]\!]^c = (Var \to U) \setminus [\![F]\!]$

$[\![F \to G]\!] = [\![\neg F \vee G]\!]$

$[\![\forall x F]\!] = \{ \eta \mid \omega \in [\![F]\!] \text{ for all } \omega : \omega = \eta \text{ except at } x \}$

$[\![\exists x F]\!] = \{ \eta \mid \omega \in [\![F]\!] \text{ for some } \omega : \omega = \eta \text{ except at } x \}$

"Everything has a value"

$F \vee \neg F$

Classical First-Order Logic

Frege 1879

Syntax $F ::= p(\bar{t}) \mid \neg F \mid F \wedge G \mid F \vee G \mid F \to G \mid \forall x\, F \mid \exists x\, F$

Semantics In an interpretation I that chooses some set U as universe assigning relation $I(p) \subseteq U^n$ to predicate symbol p/n, function $I(f) : U^n \to U$ to function symbol f/n

semantics of F is the set $[\![F]\!]$ of variable assignments $\eta : Var \to U$ in which F evaluates to true:

$[\![p(\bar{t})]\!] = \{\eta : \eta[\![\bar{t}]\!] \in I(p)\}$

$[\![\neg F]\!] = ([\![F]\!])^c$

$[\![F \wedge G]\!] = [\![F]\!] \cap [\![G]\!]$

$[\![F \vee G]\!] = [\![F]\!] \cup [\![G]\!]$

$[\![F \to G]\!] = [\![\neg F \vee G]\!]$

$[\![\forall x\, F]\!] = \{\eta : \omega \in [\![F]\!]$ for all ω with $\eta = \omega$ except at $x\}$

$[\![\exists x\, F]\!] = \{\eta : \omega \in [\![F]\!]$ for some ω — || — $\}$

A	B	$(A \to B) \vee (B \to A)$
Y	Y	Y
Y	N	Y
N	Y	Y
N	N	Y

Unification Algorithm

$$\frac{\bar{s} \doteq \bar{t} \mid \theta}{f(\bar{s}) \doteq f(\bar{t}) \mid \theta} \qquad \frac{}{(\cdot) \doteq (\cdot) \mid (\cdot)} \qquad \frac{s \doteq t \mid \theta \quad \bar{s}\theta \doteq \bar{t}\theta \mid \bar{\theta}}{(s,\bar{s}) \doteq (t,\bar{t}) \mid \theta\bar{\theta}}$$

$$\frac{x \notin FV(t)}{x \doteq t \mid (t/x)} \qquad \frac{}{x \doteq x \mid (\cdot)} \qquad \frac{x \notin FV(t),\ t = f(\bar{t})}{t \doteq x \mid (t/x)}$$

Extend Unification to FOL

Fragment suffices

$$\frac{s \doteq t \mid \theta}{p(\bar{s}) \doteq p(\bar{t}) \mid \theta}$$

$$\left(\frac{F_1 \doteq G_1 \mid \theta \qquad F_2\theta \doteq G_2\theta \mid \theta_2}{F_1 \vee F_2 \doteq G_1 \vee G_2 \mid \theta\,\theta_2} \qquad \frac{F \doteq G \mid \theta}{\forall x F \doteq \forall \underset{y}{x}\, G} \right)$$

we'll take care to avoid capture and rename as needed

Resolution / Prop

$$\frac{P \lor L \qquad \overset{P \to K}{\neg P \lor K}}{L \lor K}\ Ro$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \to P}\,id \quad \dfrac{}{P, \bot \to L \lor K}\,\bot L}{P, \neg P \to L \lor K}\,\to L \quad \dfrac{\dfrac{}{K \to K}\,id}{P, K \to L \lor K}}{P, \neg P \lor K \to L \lor K}\,\lor L \quad \dfrac{\dfrac{}{L \to L}\,id}{L, \neg P \lor K \to L \lor K}\,\lor R}{P \lor L, \neg P \lor K \to L \lor K}\,\lor L$$

$$\frac{P \lor L_1 \lor \dots \lor L_n \qquad \neg P \lor K_1 \lor \dots \lor K_m}{L_1 \lor \dots \lor L_n \lor K_1 \lor \dots \lor K_m}\ Ro$$

Resolution / FOL

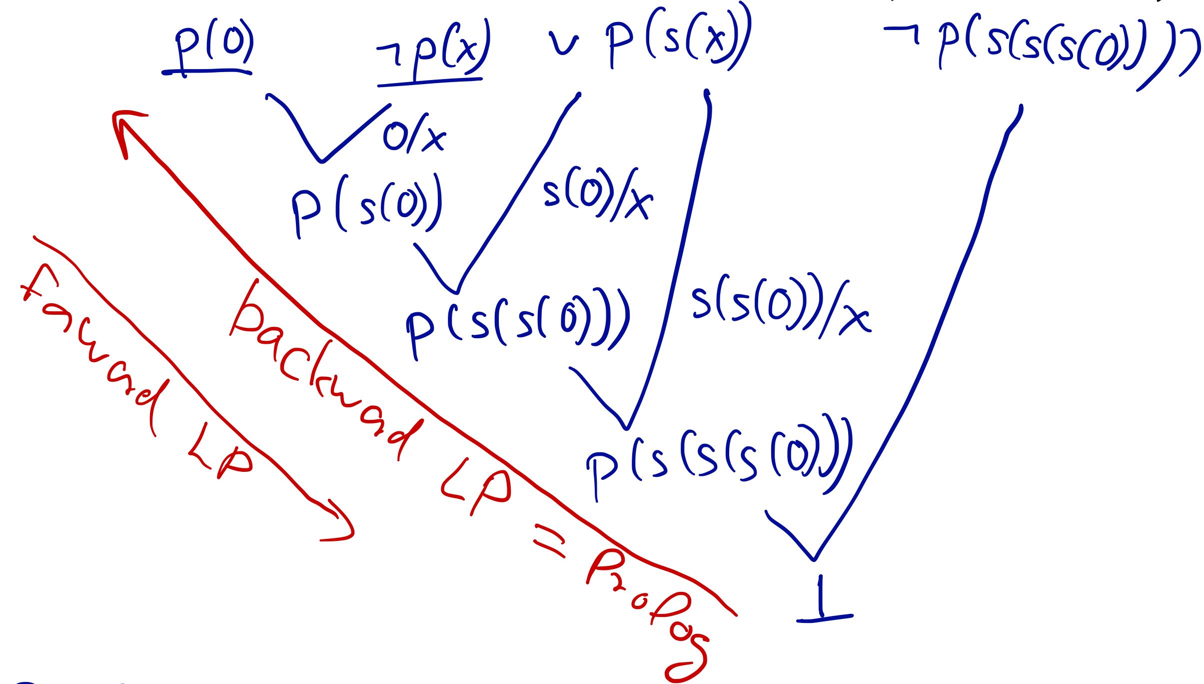
$$\frac{P \lor L \qquad \neg R \lor K}{(L \lor K)\theta}\ R \text{ if } P \doteq R \mid \theta$$

$$\frac{P \lor L_1 \lor \dots \lor L_n \qquad \neg R \lor K_1 \lor \dots \lor K_m}{(L_1 \lor \dots \lor L_n \lor K_1 \lor \dots \lor K_m)\theta}\ R \text{ if } P \doteq R \mid \theta$$

$$\frac{L_1 \lor \dots \lor L_i \lor \dots \lor L_j \lor \dots \lor L_n}{(L_1 \lor \dots \lor L_i \lor \dots \qquad \lor L_n)\theta}\ \text{Factor if } L_i \doteq L_j \mid \theta$$

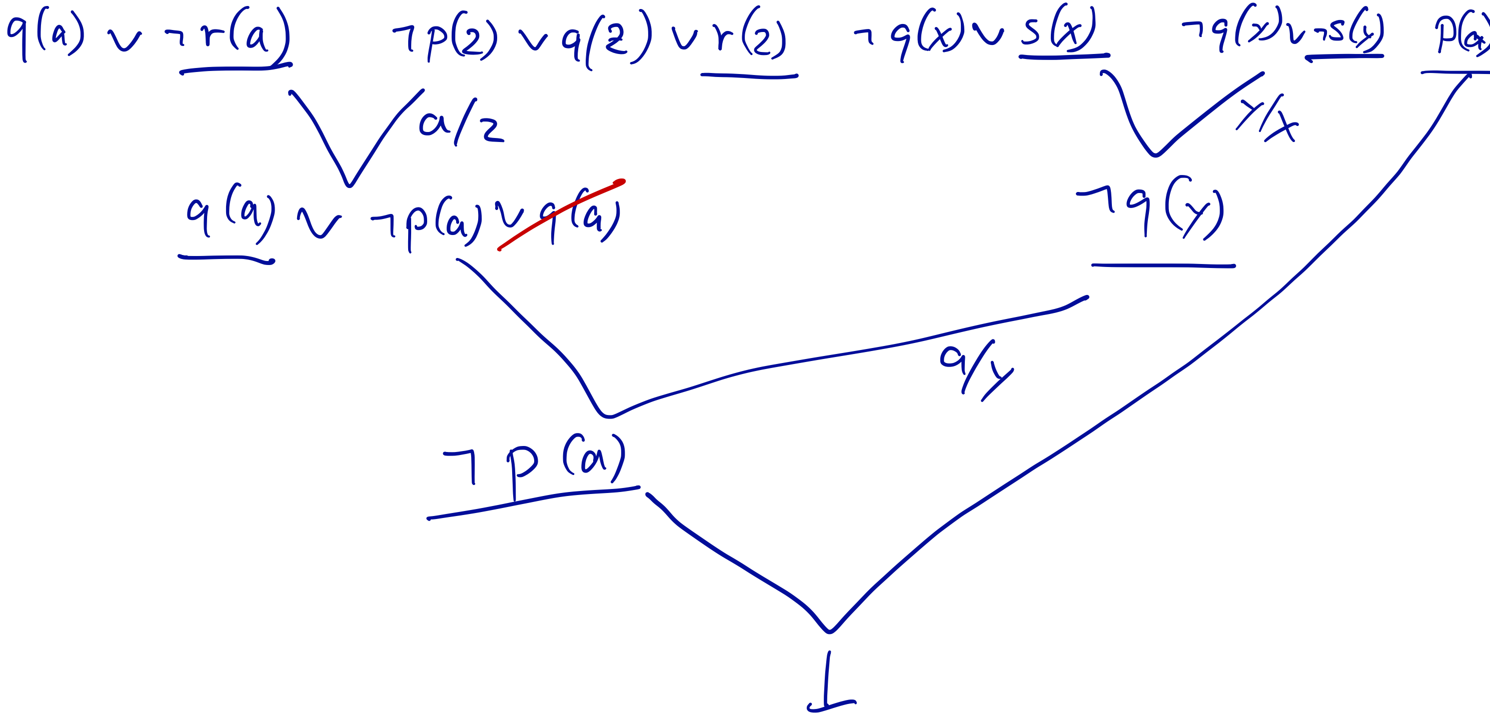
Resolution Examples

$p(0) \qquad p(s(x)) \leftarrow p(x) \qquad$ goal $?- p(s(s(s(0))))$



Prolog uses
SLD-resolution := always pure negative clause
always with input clause
selecting left most subgoal

General Resolution Example



Resolution with Factoring Example

$p(x) \vee p(y)$ $\qquad$ $\neg p(u) \vee \neg p(v)$

y/x: $p(x)$

x/u: $p(y) \vee \neg p(v)$

u/v: $\neg p(u)$

u/x: $\perp$

$P \wedge Q \mathrel{\hat{=}} \{P, Q\}$

$P \vee Q \mathrel{\hat{=}} \{P \vee Q\}$

$\forall x\, P \mathrel{\hat{=}} \{P\}$

$\exists x\, P(x) \mathrel{\hat{=}} \{P(f(y))\}$

$\forall x \exists y\, P(x,y) \mathrel{\hat{=}} \{P(x, f(x))\}$

$\exists u \forall x\, P(u,x) \mathrel{\hat{=}} \{P(a,x)\}$

$\forall x$

Resolution needs inputs renamed

$\forall x \quad p(x)$ $\qquad$ $\neg p(f(x))$

y/x

$\forall y \quad p(y)$

$f(x)/y$

$\perp$

Thm (Soundness):

Soundness of resolution follows from soundness of unification

Thm (R-Completeness):

Resolution with Factoring derives $\bot$ from M iff M unsat

$$M \vdash C$$

$$M, \neg C \vdash_R \bot$$