Quantifiers & Dependent Types?

Have: natural deduction $\top, \bot, \wedge, \vee, \supset,$ $\forall, \exists$

$\Longrightarrow$ sequent calculus $\top, \bot \wedge, \vee, \supset$

$\longrightarrow$ restricted seq. calc. $\top, \bot, \wedge, \vee, \supset$

cut elimination $\top, \bot, \wedge, \vee, \supset$

computional interpret

Quantifiers / & Dependent Types

c is new

Σ cannot have dups

$$\dfrac{\begin{array}{c}\overline{c:\tau}\\ \vdots \\ A(c)\uparrow\end{array}}{\forall x{:}\tau.\, A(x)\uparrow}\ \forall I^{c}$$

Γ, RHS only mention Σ

$$\dfrac{\Sigma, c{:}\tau;\ \Gamma \Longrightarrow A(c)}{\Sigma;\ \Gamma \Longrightarrow \forall x{:}\tau.\, A(x)}\ \forall R$$

$$\dfrac{\forall x{:}\tau.\, A(x)\downarrow \qquad t:\tau}{A(t)\downarrow}\ \forall E$$

$$\dfrac{\Sigma \vdash t:\tau \qquad \Sigma;\ \Gamma, \forall x{:}\tau.\, A(x), A(t) \Longrightarrow C}{\Sigma;\ \Gamma, \forall x{:}\tau.\, A(x) \Longrightarrow C}\ \forall L$$

$$\dfrac{t:\tau \qquad A(t)\uparrow}{\exists x{:}\tau.\, A(x)\uparrow}\ \exists I$$

$$\dfrac{\Sigma \vdash t:\tau \qquad \Sigma;\ \Gamma \Longrightarrow A(t)}{\Sigma;\ \Gamma \Longrightarrow \exists x{:}\tau.\, A(x)}\ \exists R$$

t is arbitrary

$$\dfrac{\exists x{:}\tau.\, A(x)\downarrow \qquad \begin{array}{c}\overline{c:\tau}\quad \overline{A(c)\downarrow}^{\,u}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\ \exists E^{c,u}$$

$$\dfrac{\Sigma, c{:}\tau;\ \Gamma, \exists x{:}\tau.\, A(x), A(c) \Longrightarrow C}{\Sigma;\ \Gamma, \exists x{:}\tau.\, A(x) \Longrightarrow C}\ \exists L$$

Theorem (Cut): If $\Sigma;\Gamma \overset{\mathcal{D}}{\Longrightarrow} A$ and $\Sigma;\Gamma, A \overset{\mathcal{E}}{\Longrightarrow} \mathcal{C}$ then $\Sigma;\Gamma \Longrightarrow \mathcal{C}$

Lemma (Para subst): If $\Sigma \vdash t:\tau$ and $\Sigma, c:\tau;\Gamma \Longrightarrow \mathcal{C}$ then $\Sigma;[t/c]\Gamma \Longrightarrow [t/c]\mathcal{C}$

$$\mathcal{D} = \dfrac{\overset{\mathcal{T}}{\Sigma \vdash t:\tau} \qquad \overset{\mathcal{D}_1}{\Sigma;\Gamma \Longrightarrow A_1(t)}}{\Sigma;\Gamma \Longrightarrow \exists x:\tau.\, A_1(x)}\ \exists R \qquad \mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Sigma, c:\tau;\Gamma, \exists x:\tau.\, A_1(x), A_1(c) \Longrightarrow C}}{\Sigma;\Gamma, \exists x:\tau.\, A_1(x) \Longrightarrow C}\ \exists L$$

$\Sigma;\Gamma, \exists x:\tau.\, A_1(x), A_1(t) \Longrightarrow C$ by SP$(\mathcal{E}_1\,[t/c])$

$\Sigma;\Gamma, A_1(t) \Longrightarrow C$ by IH $\exists x.\, A_1(x)$, $\mathcal{D}$, $\mathcal{E}_1$ above

$\Sigma;\Gamma \Longrightarrow C$ by IH ... $A_1(t)\{\exists x.\, A_1(x)\}$, $\mathcal{D}_1$, above

Theorem (Cut) If $\Sigma;\Gamma \xRightarrow{\mathcal{D}} A$ and $\Sigma;\Gamma, A \xRightarrow{\mathcal{E}} C$ then $\Sigma;\Gamma \Rightarrow C$

Lemma (Par. subst): If $\Sigma \vdash t:\tau$ and $\Sigma, c:\tau;\Gamma \Rightarrow A$ then $\Sigma;[t/c]\Gamma \Rightarrow [t/c]A$

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Sigma, c:\tau;\Gamma \Rightarrow A_1(c)}}{\Sigma;\Gamma \Rightarrow \forall x:\tau.\,A_1(x)}\,\forall R \qquad \mathcal{E} = \dfrac{\overset{\mathcal{T}}{\Sigma \vdash t:\tau} \quad \overset{\mathcal{E}_1}{\Sigma;\Gamma, \forall x:\tau.\,A_1(x), A_1(t) \Rightarrow C}}{\Sigma;\Gamma, \forall x:\tau.\,A_1(x) \Rightarrow C}\,\forall L$$

$\Sigma;\Gamma \Rightarrow A_1(t)$ by $SP(\mathcal{D}_1, [t/c])$

$\Sigma;\Gamma, A_1(t) \Rightarrow C$ by IH $\forall x:\tau.\,A_1(x), \mathcal{D}, \mathcal{E}_1 < \mathcal{E}$

$\Sigma;\Gamma \Rightarrow C$ by IH $A_1(t) < \forall x\, A_1(x)$, $\cdot$

Restricted Sequent Calculus

$$\frac{\Sigma, c:\tau; \Gamma \longrightarrow A(c)}{\Sigma; \Gamma \longrightarrow \forall x{:}\tau.\, A(x)}\ \forall R \qquad \frac{\Sigma \vdash t:\tau \quad \Sigma; \Gamma, \forall x{:}\tau.\, A(x), A(t) \longrightarrow C}{\Sigma; \Gamma, \forall x{:}\tau.\, A(x) \longrightarrow C}\ \forall L$$

$$\frac{\Sigma \vdash t:\tau \quad \Sigma; \Gamma \longrightarrow A(t)}{\Sigma; \Gamma \longrightarrow \exists x{:}\tau.\, A(x)}\ \exists R \qquad \frac{\Sigma, c:\tau; \Gamma, \ldots, A(c) \longrightarrow C}{\Sigma; \Gamma, \exists x{:}\tau.\, A(x) \longrightarrow C}\ \exists L$$

Uni-typed calculus

Existence Presupp $\exists x\, \top$

0 is a term

nonempty types

chew

$$\frac{\Gamma \longrightarrow A(c)}{\Gamma \longrightarrow \forall x\; A(x)}\ \forall R$$

$$\frac{\Gamma, \forall x\, .\, A(x), A(t) \longrightarrow C}{\Gamma, \forall x\, .\, A(x) \longrightarrow C}\ \forall L$$

$$\frac{\Gamma \longrightarrow A(t)}{\Gamma \longrightarrow \exists x\, .\, A(x)}\ \exists R$$

$$\frac{\Gamma, A(c) \longrightarrow C}{\Gamma, \exists x\, .\, A(x) \longrightarrow C}\ \exists L$$

chew

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\, A(x), A(t) \longrightarrow A(t)}\ \text{init}}{\forall x\, A(x), A(t) \longrightarrow \exists x\; A(x)}\ \exists R}{\forall x\, A(x) \longrightarrow \exists x\; A(x)}\ \forall L}{\longrightarrow \forall x\; A(x) \supset \exists x\; A(x)}\ \supset R$$

Computational interpretation / Proof terms

$$\dfrac{\dfrac{\overline{c:\tau}}{\vdots}\\ M : A(c)}{\lambda c. M : \forall x{:}\tau.\, A(x)}\ \forall I^{c}$$

$$\dfrac{M : \forall x{:}\tau.\, A(x) \qquad t:\tau}{M\,t : A(t)}\ \forall E$$

$$\dfrac{t:\tau \qquad M : A(t)}{\langle t, M\rangle : \exists x{:}\tau.\, A(x)}\ \exists I$$

$$\dfrac{M : \exists x{:}\tau.\, A(x) \qquad \dfrac{\overline{c:\tau}\quad \overline{u : A(c)}^{\,u}}{\vdots}\\ N : C}{\text{let } \langle c, u\rangle = M \text{ in } N : C}\ \exists E^{c,u}$$

dependent function type

$\Pi\, n{:}nat.\ vector(n)$

$\Pi\, n,m,l{:}nat.\ (matrix(n,m) \wedge matrix(m,l) \supset matrix(n,l))$

dependent sum/pair type

$\Sigma\, n{:}nat.\ vector(n)$

vector with promise of carrying its dimension

$\langle n, vecl\rangle$

$nat \times vector$

$\langle 20, (5, 0)\rangle$

$\supset$	$\to$
$\wedge$	$\langle\ \rangle$
$\vee$	case
$\forall$	Π
$\exists$	Σ