

Lecture Notes on Classical Logic

15-317: Constructive Logic
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1 Introduction

In this lecture, we design a judgmental formulation of classical logic. To gain an intuition, we explore various equivalent notions of the essence of classical reasoning including the Law of the Excluded Middle and Double-Negation Elimination. Throughout the discussion a common theme is the indirectness and “dishonesty” of classical proofs. Eventually, we arrive at a judgmentally parsimonious system based on the principle of Proof by Contradiction and founded on two new forms of judgment: *A is false* (written $A \text{ false}$) and *contradiction* (written $\#$).

2 Example

Classical reasoning is pervasive in classical mathematics, so we begin with a typical example of a theorem proven using classical methods.

Theorem: $\exists a, b \in \mathbb{R}. \text{irrational}(a) \wedge \text{irrational}(b) \wedge \text{rational}(a^b)$

Proof: Consider $\sqrt{2}^{\sqrt{2}}$: this number is either rational or irrational. Suppose it is rational: then $a = \sqrt{2}, b = \sqrt{2}$ gives the required result. Suppose it is not: then $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ gives the required result, as $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = (\sqrt{2})^2 = 2$.

*Edits by Giselle Reis

Although this claims to be a proof of an existential theorem, it gives the reader no grasp on what numbers actually witness the result—it offers two possible choices but no further guidance as to which one is correct.

A slightly whimsical characterization of the proof offers a glimpse into the computational content of classical reasoning. Imagine a prominent mathematician delivering the above proof aloud as part of a lecture before a large audience. Initially, he delivers only the first half, saying, “Let $a = \sqrt{2}$ and $b = \sqrt{2}$; then a and b are irrational while a^b is rational—*trust me*.” Amidst some mumbling, the audience nods and accepts the proof; after all, he *is* a very prominent mathematician, and he probably knows what he’s talking about. But then, halfway through the lecture, a student from the back suddenly leaps up and exclaims, “His proof is no good—I have a proof that $\sqrt{2}^{\sqrt{2}}$ is *irrational!*”¹. The audience gasps and a murmur runs through the crowd, but before anyone else can speak, the mathematician calmly responds, “May I see your proof?” After checking it over, the mathematician addresses the crowd again: “My apologies—I misspoke earlier. What I meant to say was this: Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$; then a and b are irrational—I have this proof, if you don’t believe me!—while $a^b = 2$ and is therefore rational.”

The poor student at the back thought she could attain fame and fortune by debunking the prominent mathematician, but in fact, the mathematician stole the glory by leveraging her proof for his own ends. Classical proofs exhibit a similar time-traveling behavior when executed, as we’ll see in the next lecture.

3 What is classical logic?

Classical logic can be characterized by a number of equivalent axioms:

- Proof by Contradiction
- Law of the Excluded Middle: $A \vee \neg A$
- Double-Negation Elimination: $\neg\neg A \supset A$
- Peirce’s Law: $((A \supset B) \supset A) \supset A$

¹It is, in fact, though the proof is non-trivial.

If we want to allow the kinds of proofs that we've seen before in our logic, then we need to make it classical. We can do this by adding one or more rules that correspond to these axioms. For instance, we might add:

$$\frac{}{A \vee \neg A \text{ true}} \text{LEM} \quad \frac{\neg\neg A \text{ true}}{A \text{ true}} \text{DNE}$$

But adding both rules is redundant. It turns out that *DNE* is a derived rule in the system of natural deduction (for intuitionistic logic) + *LEM*. Analogously, *LEM* is a derived rule in the system of natural deduction + *DNE*. When a rule can be derived in a system, it means that it is "useless"², i.e., the system has the same power with and without it. This is because in every derivation where the derived rule is used, it can be replaced by a sequence of other rules having the same effect.

To show that only one of the rules above is enough, we will show that we can use the rules of natural deduction + one to derive the other. First, let's show how we can derive *DNE* from *LEM*:

$$\frac{\frac{\frac{}{A \vee \neg A \text{ true}} \text{LEM} \quad \frac{}{A \text{ true}} u \quad \frac{\frac{\frac{}{\neg\neg A \text{ true}} \quad \frac{}{\neg A \text{ true}} v}{\perp \text{ true}} \supset E}{A \text{ true}} \perp E}{\vee E^{u,v}}}{A \text{ true}} \text{DNE}}$$

As you can see, it is precisely the power of having the middle excluded that lets us turn our proof of $\neg\neg A \text{ true}$ into a proof of $\perp \text{ true}$, and thus a proof of $A \text{ true}$ as required.

Using *DNE* also allows us to derive *LEM*. Note that since the *LEM* rule has no premises, this will have to be a completely closed derivation. To help elucidate the thought process of a classical prover, we'll do a step-by-step derivation. We start bottom up from the conclusion:

$$\begin{array}{c} \vdots \\ A \vee \neg A \text{ true} \end{array}$$

If we were to attempt to proceed as we've done previously, we would now have to apply a disjunction-introduction rule, but we have no way of deciding which injection to choose: merely positing all propositions to be true or not does nothing to make apparent which is the case. We know

²Sometimes it is useful to abbreviate proofs, for example. But derived rules should be used parcimoniously, since having too many rules is bad for proof search.

that this classical tautology is unprovable intuitionistically, anyhow, so we have no hope but to begin by employing our classical rule, double-negation elimination.

$$\frac{\begin{array}{c} \vdots \\ \neg\neg(A \vee \neg A) \text{ true} \end{array}}{A \vee \neg A \text{ true}} \text{ DNE}$$

Since $\neg A$ is notationally defined to mean $A \supset \perp$, we can take the usual no-brainer step of applying implication introduction:

$$\frac{\begin{array}{c} \frac{}{\neg(A \vee \neg A) \text{ true}} \text{ } u \\ \vdots \\ \perp \text{ true} \end{array}}{\neg\neg(A \vee \neg A) \text{ true}} \supset I^u \\ \frac{}{A \vee \neg A \text{ true}} \text{ DNE}$$

Now we are tasked with proving falsehood true: a difficult task, to be sure, but a glimmer of hope shines through in our sole assumption u . Perhaps by applying it to an appropriate argument, we can conclude $\perp \text{ true}$ as required:

$$\frac{\begin{array}{c} \frac{}{\neg(A \vee \neg A) \text{ true}} \text{ } u \\ \vdots \\ \perp \text{ true} \end{array}}{\neg\neg(A \vee \neg A) \text{ true}} \supset I^u \\ \frac{\frac{}{\neg(A \vee \neg A) \text{ true}} \text{ } u \quad \frac{}{A \vee \neg A \text{ true}} \text{ } u}{\perp \text{ true}} \supset E \\ \frac{}{A \vee \neg A \text{ true}} \text{ DNE}$$

At this point, if we were not being careful, we might throw up our hands and quit. We're right back to where we started, trying to prove $A \vee \neg A \text{ true}$! But now we have an extra hypothesis to help us. Undaunted, we pause to consider: which shall we prove, A or $\neg A$? Since we know nothing of the structure of A , we have no hope of proving it unless an assumption can yield it, but nothing seems appropriate. So instead, we try for the right disjunct, $\neg A$. We can also go ahead and apply implication introduction without thinking twice, this time attaining an assumption v that $A \text{ true}$.

$$\begin{array}{c}
 \frac{}{\neg(A \vee \neg A) \text{ true}}^u, \frac{}{A \text{ true}}^v \\
 \vdots \\
 \frac{\perp \text{ true}}{\neg A \text{ true}} \supset I^v \\
 \frac{}{\neg(A \vee \neg A) \text{ true}}^u \quad \frac{\perp \text{ true}}{\neg A \text{ true}} \supset I^v \\
 \frac{}{A \vee \neg A \text{ true}} \vee I_R \\
 \frac{}{\perp \text{ true}} \supset E \\
 \frac{}{\neg\neg(A \vee \neg A) \text{ true}} \supset I^u \\
 \frac{}{A \vee \neg A \text{ true}} DNE
 \end{array}$$

This situation is familiar once again: to prove $\perp \text{ true}$, we must use an assumption, and u still seems to be the only one that can help. So we carry out another implication elimination, feeling only the slightest sense of *deja vu*...

$$\begin{array}{c}
 \frac{}{\neg(A \vee \neg A) \text{ true}}^u, \frac{}{A \text{ true}}^v \\
 \vdots \\
 \frac{}{\neg(A \vee \neg A) \text{ true}}^u \quad \frac{}{A \vee \neg A \text{ true}} \\
 \frac{}{\perp \text{ true}} \supset E \\
 \frac{}{\neg(A \vee \neg A) \text{ true}}^u \quad \frac{\perp \text{ true}}{\neg A \text{ true}} \supset I^v \\
 \frac{}{A \vee \neg A \text{ true}} \vee I_R \\
 \frac{}{\perp \text{ true}} \supset E \\
 \frac{}{\neg\neg(A \vee \neg A) \text{ true}} \supset I^u \\
 \frac{}{A \vee \neg A \text{ true}} DNE
 \end{array}$$

Once more we are faced with our great adversary, the *tertium non datur*³ himself! But now the tables are turned: no longer must we cower in fear behind our security blanket of negation. No, now we can prove $A \text{ true}$ itself, by the very assumption we hypothesized to prove $\neg A \text{ true}$! The derivation is now complete.

³Latin for “no third (option) is given”.

$$\begin{array}{c}
 \frac{\frac{\frac{}{\neg(A \vee \neg A) \text{ true}} u \quad \frac{\frac{\frac{}{A \text{ true}} v}{A \vee \neg A \text{ true}} \vee I_L}{A \vee \neg A \text{ true}} \supset E}{\perp \text{ true}} \supset I^v}{\neg A \text{ true}} \supset I^v}{\frac{\frac{}{\neg(A \vee \neg A) \text{ true}} u \quad \frac{\frac{\frac{}{A \vee \neg A \text{ true}} \vee I_R}{A \vee \neg A \text{ true}} \supset E}{\perp \text{ true}} \supset I^u}{\neg \neg(A \vee \neg A) \text{ true}} \supset I^u}{A \vee \neg A \text{ true}} DNE}
 \end{array}$$

An interesting point to note about this proof is that, save for the last line, it is a perfectly valid intuitionistic proof. Only at the very end—or beginning, as we’ve told the story—did we need to appeal to classical reasoning in the form of *DNE*. In fact, this observation hints at a more general one: there is a *double-negation translation* that translates any classical theorem into one that is intuitionistic valid.

Although it should be clear that adding either of *LEM* or *DNE* would suffice to make our logic classical, such rules violate the aesthetic principles we’ve adhered to thus far: both rules contain connectives, but they are neither introducing nor eliminating any single connective, and both rules contain multiple instances of a connective, suggesting a certain non-orthogonality of principles. Must our logic include negation and disjunction in order to become classical? Perhaps just negation? Or, since negation as we’ve seen it has been defined as implying falsehood, perhaps implication and falsehood are the important characteristics.

In fact, we can characterize what it means for a logic to be classical without appealing to any connectives at all, using purely judgmental notions. This is what we shall now endeavor to do.

4 Towards a better proof theory

Our proof theory for classical logic will be based on the idea of proof by contradiction. Proof by contradiction is closely related to double-negation elimination: If we take the rule *DNE* and require that its premise be proven by implication-introduction (as we know we may), we find that we could replace it by a rule that looks like this:

$$\frac{\overline{\neg A \text{ true}}^u}{\vdots} \frac{\perp \text{ true}}{A \text{ true}} ?PBC^u$$

If an assumption that $\neg A$ is true can lead to a proof of falsehood, we may conclude (classically, anyhow) that A must be true.

Proof by Contradiction. But we should keep in mind the design principles that keep our logics clear and easy to reason about: at most one connective should occur in a given rule, with all other notions being at the level of judgments. The proposed rule above still contains two connectives, \neg and \perp , and does not read like an introduction or elimination rule, so we would like to replace it with a rule that appeals only to judgmental notions. We achieve this by inventing two new judgment forms, $A \text{ false}$ and $\#$ (contradiction):

$$\frac{\overline{A \text{ false}}^k}{\vdots} \frac{\#}{A \text{ true}} PBC^k$$

By convention we use letters like k and ℓ to denote hypotheses of falsity.

Now, of course, we must explain the meaning of the new judgment forms $A \text{ false}$ and $\#$. We understand these judgments through certain principles analogous to the Substitution Principle we posited for hypotheses of the form $A \text{ true}$ before. First, we require that contradiction yield anything:

$\mathcal{D} \quad \mathcal{D}$
Contradiction Principle: If $\#$, then J .

In this principle, J stands for any judgment.

Judgments of Falsehood. The *false* judgment is treated somewhat specially: we derive its meaning from the meaning of *true*. We take $A \text{ false}$ as a conclusion to be a judgment-level notational definition:

$$A \text{ false} := \begin{array}{c} A \text{ true} \\ \vdots \\ \# \end{array}$$

To prove a proposition false, we assume it true and derive a contradiction.

We still require hypotheses of the form $A \text{ false}$, since they appear in our classical rule PBC^k . To explain the meaning of hypotheses of $A \text{ false}$, we derive a substitution principle from the definition above.

$$\frac{}{A \text{ true}} u \quad \frac{}{A \text{ false}} k \quad \frac{D}{A \text{ false}} k$$

Falsity Substitution Principle: If # and J then J

The final deduction in the above no longer depends on the hypothesis k nor on the hypothesis u (indicated by the double-rule notation). If the hypothesis u that $A \text{ true}$ leads to a contradiction by proof \mathcal{D} and if the hypothesis k that $A \text{ false}$ leads to J by proof \mathcal{E} , then J has a more direct justification where \mathcal{D} is used as evidence for the hypothesis $A \text{ false}$ of the proof \mathcal{E} .

There's an important difference between this substitution principle and the one for true hypotheses: we took that substitution principle as a given, since it arose directly from the meaning of the hypothetical judgment. Here, though, since we did not directly define $A \text{ false}$ as a conclusion, we are not substituting proofs that conclude $A \text{ false}$ for hypotheses of $A \text{ false}$. Therefore, this substitution principle is one we must prove of our inference rules, and we use the double-ruled notation to denote the operation embodied by this proof. (An alternative notation is $\llbracket u. \mathcal{D}/k \rrbracket \mathcal{E}$.) We defer any further discussion until later when we learn rule induction.

Contradictions. We have one more thing to define before the system is complete, and that is how to derive contradiction! With the rules given above, there are currently no ways to do so. Contradiction means that some proposition is both true and false, so we take the following rule:

$$\frac{A \text{ false} \quad A \text{ true}}{J} \text{ contra}$$

Note that we do not suppose the rule to conclude # directly, since we wish to ensure that the Contradiction Principle above remains true. Instead, we allow the rule to conclude any judgment J , including #.

This concludes the discussion of the judgments $A \text{ false}$ and #. There is one remaining important technical matter to deal with, though: since

we have introduced one new judgment that may appear as a conclusion, #, we must revisit any rules we previously defined that were supposed to conclude an arbitrary conclusion. In particular, the rules $\vee E^{u,v}$ and $\perp E$ must be generalized to allow a conclusion of any judgment, and not merely one of the form $C \text{ true}$ ⁴:

$$\frac{\frac{\frac{\overline{A \text{ true}}^u}{J} \quad \frac{\overline{B \text{ true}}^v}{J}}{\vee E^{u,v}} \quad \frac{\overline{\perp \text{ true}}}{J} \perp E}{J} A \vee B \text{ true}$$

To show an example of these rules in action, we derive Peirce’s Law, one of the characterizations of classical logic mentioned above. We show only the final deduction, but it has a flavor very similar to the derivation of $A \vee \neg A$ from *DNE*.

$$\frac{\frac{\overline{A \text{ false}}^k \quad \frac{\frac{\overline{A \text{ true}}^v}{\text{contra}}}{B \text{ true}} \supset I^v}{A \supset B \text{ true}} \supset E}{\frac{\overline{A \text{ false}}^k \quad \frac{\overline{(A \supset B) \supset A \text{ true}}^u}{A \text{ true}}}{\text{contra}}}{\frac{\overline{A \text{ true}}}{\text{contra}} \supset I^u} \# \text{ PBC}^k} ((A \supset B) \supset A) \supset A \text{ true}$$

Note the two uses of the hypotheses k that $A \text{ false}$, which is similar to the two uses of the refutation of $A \vee \neg A$ in the derivation of *LEM* above. Notice also the two uses of the rule *contra* to conclude two different judgments.

5 Proofs and truths

One can say that classical logic is a logic of *truths*. Indeed, you can even find out the value of a proposition with the use of truth tables, remember? They can be used to show, for example, that $A \supset B \equiv \neg A \vee B$.

Intuitionistic logic is a logic of *proofs*, where a proposition is only considered valid if there is a proof of it. No truth tables allowed! So if we want

⁴We would also have to similarly generalize the existential elimination rule, $\exists E^a$, but in the present lecture we treat only the propositional fragment.

6 Negation Revisited

Armed with our new judgmental notions, we can now give a direct definition of negation rather than one that simply defines it in terms of implication and falsehood. The new rules are as follows:

$$\frac{\frac{\frac{}{A \text{ true}} u}{\vdots} \#}{\neg A \text{ true}} \neg I^u \quad \frac{\frac{\frac{}{A \text{ false}} k}{\vdots} J}{\neg A \text{ true}} \neg E^k$$

The introduction rule essentially amounts to saying, “ $\neg A$ true if A false,” but replacing A false with its judgmental definition. The elimination rule lets you extract A false from a proof of $\neg A$ true, but since we only allow A false as a hypothesis, the rule introduces a new scope with an extra assumption rather than simply concluding A false directly.

This elimination rule is locally sound, as witnessed by the following local reduction, which makes use of the Falsity Substitution Principle:

$$\frac{\frac{\frac{\frac{}{A \text{ true}} u}{\mathcal{D}} \#}{\neg A \text{ true}} \neg I^u \quad \frac{\frac{\frac{}{A \text{ false}} k}{\mathcal{E}} J}{\neg E^k}}{J} \implies_R \frac{\frac{\mathcal{D}}{A \text{ false}} k}{\mathcal{E}} J$$

It is also locally complete; we can expand an arbitrary deduction of $\neg A$ true in two ways:

$$\frac{\mathcal{D}}{\neg A \text{ true}} \implies_E \frac{\frac{\mathcal{D}}{\neg A \text{ true}} \quad \frac{\frac{\frac{}{A \text{ false}} k}{\#} \neg I^u}{\neg A \text{ true}} \neg E^k}{\neg A \text{ true}} \text{contra}$$

$$\frac{\mathcal{D}}{\neg A \text{ true}} \implies_E \frac{\frac{\frac{\mathcal{D}}{\neg A \text{ true}} \quad \frac{\frac{}{A \text{ true}} u}{\#} \neg I^u}{\neg A \text{ true}} \neg E^k}{\neg A \text{ true}} \text{contra}$$

Judgmental Definitions

$$A \text{ false} := \begin{array}{c} A \text{ true} \\ \vdots \\ \# \end{array} \quad \frac{A \text{ false} \quad A \text{ true}}{J} \text{ contra}$$

Classical Rule

$$\frac{\begin{array}{c} \overline{\quad}^k \\ A \text{ false} \\ \vdots \\ \# \end{array}}{A \text{ true}} PBC^k$$

Changed Rules

$$\frac{\begin{array}{c} \overline{\quad}^u \quad \overline{\quad}^v \\ A \text{ true} \quad B \text{ true} \\ \vdots \quad \vdots \\ A \vee B \text{ true} \quad J \quad J \end{array}}{J} \vee E^{u,v} \quad \frac{\perp \text{ true}}{J} \perp E$$

Rules for Negation

$$\frac{\begin{array}{c} \overline{\quad}^u \\ A \text{ true} \\ \vdots \\ \# \end{array}}{\neg A \text{ true}} \neg I^u \quad \frac{\begin{array}{c} \overline{\quad}^k \\ A \text{ false} \\ \vdots \\ \neg A \text{ true} \quad J \end{array}}{J} \neg E^k$$

Figure 1: Rules for classical natural deduction