

Some people, perhaps overly aggressively, call intuitionistic logic “the logic of truth” and classical logic “the logic of lies”. The two camps will never see eye-to-eye. But everyone agrees that linear logic is “the logic of food”.

1 Substructural logics

Linear logic is an example of a *substructural* logic. The sequent calculus we’re used to has a bunch of structural properties, e.g. weakening, contraction, exchange, and associativity. Substructural logics lack one or more of these properties. Today, we’re talking about a logic where *weakening* and *contraction* are restricted.

1.1 Aside: the status of identity

On the midterm, many people removed a sequent calculus rule and then said something along the lines of “now (thing) is not provable except by using identity”. This isn’t actually true, because identity is *not* a structural property! It’s a *theorem* we prove about our systems (by induction over the rules). If you remove a rule, identity probably doesn’t hold anymore!

2 Linear logic review

Linear logic has more connectives than standard intuitionistic logic! In general, as you remove structural rules, you “discover” more connectives.

Recall that we divide linear logic connectives into 3 kinds:

1. multiplicative: divide the context
2. additive: don’t divide the context
3. exponential: allow for “nonlinear” reasoning in linear logic

2.1 Conjunction

Consider the \wedge connective in normal sequent calculus. You might be bored and consider writing its right rule in 2 different ways:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \quad \frac{\Delta \Rightarrow A \quad \Gamma \Rightarrow B}{\Delta, \Gamma \Rightarrow A \wedge B}$$

These are equivalent — given the second rule, the first is admissible because

$$\frac{\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma, \Gamma \Rightarrow A \wedge B}}{\Gamma \Rightarrow A \wedge B} \text{ contraction}$$

And given the first rule, the second is admissible because

$$\frac{\frac{\dots \Delta \Rightarrow A \dots \text{weakening}}{\Delta, \Gamma \Rightarrow A} \quad \frac{\dots \Gamma \Rightarrow B \dots \text{weakening}}{\Delta, \Gamma \Rightarrow B}}{\Delta, \Gamma \Rightarrow A \wedge B}$$

But we used... contraction... and weakening... what if we don’t? Then suddenly our silly exercise of writing the rule 2 ways isn’t so pointless after all!

2.1.1 Simultaneous conjunction

We write this with \otimes and read it “tensor”. It corresponds to the second right rule for conjunction above. It’s *multiplicative*, because the context is divided.

$$\frac{\Gamma_1 \Vdash A \quad \Gamma_2 \Vdash B}{\Gamma_1, \Gamma_2 \Vdash A \otimes B} \otimes R \quad \frac{\Gamma, A, B \Vdash C}{\Gamma, A \otimes B \Vdash C} \otimes L$$

The idea with this connective is that you get to have *both* its constituents simultaneously, but that means that in order to construct one, you have to be able to construct *both* its constituents by dividing up your resources somehow.

2.1.2 Alternative conjunction

We write this with $\&$ and read it “with”. It corresponds to the first right rule for conjunction above. It’s *additive*, because you don’t divide the context.

$$\frac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \& B} \& R \quad \frac{\Gamma, A \Vdash C}{\Gamma, A \& B \Vdash C} \& L_1 \quad \frac{\Gamma, B \Vdash C}{\Gamma, A \& B \Vdash C} \& L_2$$

This is also known as “internal choice”, the idea being that *you* get to pick which one of the two constituents you’d rather have.

2.2 Disjunction

2.2.1 External choice

We write this with \oplus (I don’t know how to read it; maybe “plus”). It’s *additive*¹.

$$\frac{\Gamma \Vdash A}{\Gamma \Vdash A \oplus B} \oplus R_1 \quad \frac{\Gamma \Vdash B}{\Gamma \Vdash A \oplus B} \oplus R_2 \quad \frac{\Gamma, A \Vdash C \quad \Gamma, B \Vdash C}{\Gamma, A \oplus B \Vdash C} \oplus L$$

The left rule shows why it’s *external* choice — to make use of a \oplus , you have to account for either of the constituents being the one that’s true, because you don’t know which one will turn out to be true.

2.3 Implication

We write this with \multimap and read it “lolli” (no, really). It’s *multiplicative*. Pay careful attention to how it’s different from \supset .

$$\frac{\Gamma, A \Vdash B}{\Gamma \Vdash A \multimap B} \multimap R \quad \frac{\Gamma_1 \Vdash A \quad \Gamma_2, B \Vdash C}{\Gamma_1, \Gamma_2, A \multimap B \Vdash C} \multimap L$$

Notice the “resource consumption” going on in the left rule.

2.4 Others

We also have \top , $\mathbf{1}$, $\mathbf{0}$, and $!$.

3 The main course

As promised, we now demonstrate why linear logic is the logic of food!

¹That’s why we use the circled plus, of course.²

²That was a joke.

How would you represent the prix fixe menu³ in linear logic?

menu = \$20 \multimap appetizer \otimes entree \otimes beverage
appetizer = bamboo shoots \oplus brinjal \oplus matsutake mushrooms \oplus pickled turnip
entree = sashimi platter $\&$ (unaju \otimes ikura chawanmushi $\&$ (saba shioyaki \otimes salmon teriyaki))
beverage = (green tea $\&$ hojicha $\&$ genmaicha) \otimes ((\$15 \multimap alcohol) $\&$ **1**)
alcohol = sake $\&$ shochu

Now pretend the *Restaurant japonais du centre Gates-Hillman* is actually Olive Garden and you get unlimited free breadsticks! How would you add *that* to your linear logic representation?

Change menu to say

menu = \$20 \multimap appetizer \otimes entree \otimes beverage \otimes !breadsticks

4 Questions

- 1 Prove $\vdash A \multimap B \multimap A$.
- 2 Prove $\vdash A \& \top \multimap A$ and $\vdash A \multimap A \& \top$.
- 3 Prove $\vdash A \otimes \mathbf{1} \multimap A$ and $\vdash A \multimap A \otimes \mathbf{1}$.
- 4 Prove $\vdash (A \otimes B \multimap C) \multimap (A \multimap B \multimap C)$.

³Sorry about the menu, I don't actually know French/what a fancy Japanese restaurant menu looks like



Restaurant japonais du centre Gates-Hillman
Prix fixe menu (\$ 40)
2 décembre, 2015

Appetizer

Seasonal vegetable tempura (Bamboo shoots / brinjal / matsutake mushrooms / pickled turnip)

Choice of entrée

Sashimi platter

Unaju with ikura chawanmushi

Saba shioyaki with salmon teriyaki

Choice of beverages

Green tea, hojicha, or genmaicha

Sake / Shochu

additional \$15

