Constructive Logic (15-317), Fall 2015
Recitation 3: Proof Terms & Verification (9/16/15)
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Part I

Proof Terms

Reduce the below proof terms taking care to not capture any substitutions!

1. $(\lambda u : (A \supset B) \wedge (B \supset C).\lambda w : A.(\texttt{snd}\ u)((\texttt{fst}\ u)w))\langle(\lambda x : A.x), \lambda y : B.y)\rangle$
   $\Rightarrow_R \lambda w : A.(\texttt{snd}\ \langle(\lambda x : A.x), (\lambda y : B.y)\rangle)((\texttt{fst}\ \langle(\lambda x : A.x), (\lambda y : B.y)\rangle)w)$
   $\Rightarrow_R \lambda w : A.(\lambda y : B.y)((\texttt{fst}\ \langle(\lambda x : A.x), (\lambda y : B.y)\rangle)w)$
   $\Rightarrow_R \lambda w : A.(\lambda y : B.y)((\lambda x : A.x)w)$
   $\Rightarrow_R \lambda w : A.(\lambda y : B.y)w$
   $\Rightarrow_R \lambda w : A.w$

   Note: This reduction showed that reducing the composition of two identity functions reduces again to the identity function.

2. $(\lambda y : A.\lambda f : A \supset B.fy)(fy)$.

   We're renaming variables to avoid capture:
   $(\lambda z : A.\lambda g : A \supset B.gz)(fy)$
   $\Rightarrow_R \lambda g : A \supset B.g(fy)$

Part II

Tutch

```
annotated proof andComm : A & B => B & A =
begin
[ u: A & B;
snd u: B;
fst u: A;
(snd u, fst u): B & A];
fn u => (snd u, fst u): A & B => B & A;
end;

annotated proof mp: (A &  (A => B)) => B =
begin
[ x: A & (A => B);
fst x: A;
snd x: A => B;
(snd x) (fst x): B];
fn x => (snd x) (fst x) : A & (A => B) => B;
end;

annotated proof drop : A => (B => A) =
begin
[x: A;
  [y: B;
```

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  x: A];
  fn y => x: B => A];
fn x => fn y=> x: A => (B => A);
end;

annotated proof impDef : (~A | B) => A => B =
begin
[ x: ~A | B;
  [a:  A;
    [ na: ~A ;
      na a: F;
      abort (na a): B ];
    [ b: B ;
      b:  B ];
    case x of inl na => abort (na a) | inr b => b end: B ];
  fn a => case x of inl na => abort (na a) | inr b => b end : A => B ];
fn x => fn a => case x of inl na => abort (na a)  | inr b => b end : (~A | B) => A => B;
end;
```

Part III

Verification

1. Give two different natural deduction proofs of $A \supset A$ *true*. How many natural deduction proofs of this judgement exist?

$$\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{A \supset A\ \mathit{true}} \supset I^u \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u \quad \dfrac{}{A\ \mathit{true}}\,u}{A \wedge A\ \mathit{true}} \wedge I}{A\ \mathit{true}} \wedge E_L}{A \supset A\ \mathit{true}} \supset I^u$$

   There can be infinite proofs.

2. Give a proof of $A \supset A \uparrow$. How many proofs of this judgment exist?

$$\dfrac{\dfrac{\dfrac{}{A \downarrow}\,u}{A \uparrow} \downarrow\uparrow}{A \supset A \uparrow} \supset I^u$$

   Only one proof exists.