**Constructive Logic (15-317), Fall 2015**
**Recitation 2: Harmony and Tutch (9/9/2015)**
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## Part I
# Tutch

See the exercises in recitation2a.tut and recitation2b.tut.

A suggested set to start with:

```
proof contraction : A & A => A =
begin
[ A & A;    % brackets indicate scope of assumption A & A
  A ];      % from this assumption, we can conclude A (via and-elimination)
A & A => A; % implication-introduction
end;

proof drop : A => B => A =
begin
[ A;
  [ B;
    A ];
  B => A ];
A => B => A;
end;

proof impDef : (~A | B) => A => B =
begin
[ ~A | B;
  [ A;
    [ ~A ; F; B ];
    [ B ; B ];
    B ];
  A => B ];
(~A | B) => A => B;
end;

proof dnLem : ~~(A | ~A) =
begin
[ ~(A | ~A);
  [ A;
    A | ~A;
    F ];
  ~A;
  A | ~A;
  F ];
~~(A | ~A);
end;
```

**Part II**

# Harmony

Today we'll do an example showing how to use harmony to show that a new logical connective is well-designed.

Let's define a new unary logical connective, $\heartsuit$. It has the following introduction and elimination rules:

$$\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,w} \\ \vdots & \vdots \\ \bot\ \mathit{true} & \bot\ \mathit{true}\end{array}}{A\heartsuit B\ \mathit{true}}\ \heartsuit I^{u,w} \qquad \dfrac{A\heartsuit B\ \mathit{true} \quad A\ \mathit{true}}{\bot\ \mathit{true}}\ \heartsuit E_L \qquad \dfrac{A\heartsuit B\ \mathit{true} \quad B\ \mathit{true}}{\bot\ \mathit{true}}\ \heartsuit E_R$$

Now, we want to show soundness and completeness. It's good to have an intuitive understanding of what we're doing here: why are we doing this? Soundness means that we can't use these new rules to pull facts out of thin air. Completeness means that if we use these rules, we don't drop any facts on the floor. Let's start with soundness:

$$\dfrac{\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,u} \\ \mathcal{D} & \mathcal{E} \\ \bot\ \mathit{true} & \bot\ \mathit{true}\end{array}}{A\heartsuit B\ \mathit{true}}\ \heartsuit I^{u,w} \quad \begin{array}{c}\mathcal{F} \\ A\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\ \heartsuit E_L \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{F} \\ A\ \mathit{true} \\ \mathcal{D} \\ \bot\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{cc}\overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,u} \\ \mathcal{D} & \mathcal{E} \\ \bot\ \mathit{true} & \bot\ \mathit{true}\end{array}}{A\heartsuit B\ \mathit{true}}\ \heartsuit I^{u,w} \quad \begin{array}{c}\mathcal{F} \\ B\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\ \heartsuit E_R \quad \Rightarrow_R \quad \begin{array}{c}\mathcal{F} \\ B\ \mathit{true} \\ \mathcal{E} \\ \bot\ \mathit{true}\end{array}$$

Now, we need to show completeness:

$$\begin{array}{c}\mathcal{D} \\ A\heartsuit B\ \mathit{true}\end{array} \Rightarrow_E \dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ A\heartsuit B\ \mathit{true}\end{array} \quad \overline{A\ \mathit{true}}^{\,u}}{\bot\ \mathit{true}}\ \heartsuit E_L \quad \dfrac{\begin{array}{c}\mathcal{D} \\ A\heartsuit B\ \mathit{true}\end{array} \quad \overline{B\ \mathit{true}}^{\,w}}{\bot\ \mathit{true}}\ \heartsuit E_R}{A\heartsuit B\ \mathit{true}}\ \heartsuit I^{u,w}$$

Now, suppose we had made a mistake. Consider what would happened if we had forgotten to include rule $\heartsuit E_R$. Which would have failed: soundness or completeness? Where?

Answer: completeness would have failed, since we'd be unable to eliminate using $B\ \mathit{true}$ to get the right-hand $\bot\ \mathit{true}$ we need to complete the expansion for local completeness.

Alternatively, suppose we had defined the incorrect introduction rule below. Which proof would have failed, and why?

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ \bot\ \mathit{true}\end{array}}{A\heartsuit B\ \mathit{true}}\ \heartsuit I - oops^{u,w}$$

Answer: Soundness fails, because we won't be able to perform the second local reduction.

Another exercise: suppose we had defined $\dfrac{\begin{array}{cc}A\ \mathit{true} & B\ \mathit{true} \\ \vdots & \vdots \\ \bot\ \mathit{true} & \bot\ \mathit{true}\end{array}}{A\heartsuit B\ \mathit{true}}\ \heartsuit I$ rather than $\heartsuit I^{u,w}$. What would have gone wrong? And what if, even if we had used $\heartsuit I^{u,w}$, we had forgotten to label our hypotheses in the proof with $u$ and $w$?