Constructive Logic (15-317), Fall 2015
Recitation 1: An introduction to natural deduction (2/9/2015)
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What's a proposition? What's a judgement? What's the difference?

In lecture yesterday, there was a question about why, for $A \wedge B \supset B \wedge A$ *true*, we didn't instead write $A \wedge B$ *true* $\supset$ $B \wedge A$ *true*. This boils down to a distinction between *propositions* and *judgements*. In the words of the illustrious Frank Pfenning, "A judgement is something we may know, that is, an object of knowledge. A judgement is evident if we in fact know it... A proposition is interpreted as a set whose elements represent the proofs of the proposition."

...ok, so, what does that mean?

When talking about logic, we make a distinction between the *object language* and the *metalanguage*. The object language is the stuff we're studying, i.e. the logical connectives, the A, B, all that. The metalanguage is the language we use to *talk about* the object language, the formalisms we use to express things about our logic. *Propositions* are statements we make in the object language; they are things in the logic proper, whereas *judgements* are statements we make in the metalanguage about things in the language (like propositions!). For instance, A is a proposition whereas A *true* is a judgement that talks about that proposition.

To clarify the distinction, note that we can have more judgements than A *true*. Later in the course, we'll see plenty of these, but for now, consider this other judgement A *prop*, meaning "A is a proposition"[1]. As an example of how this judgement would be used, consider the formation of the proposition $A \wedge B$.

$$\frac{A\ \mathit{prop} \qquad B\ \mathit{prop}}{A \wedge B\ \mathit{prop}} \wedge F$$

If we went through and replaced *prop* with *true*, we would have the $\wedge I$ rule. We can have multiple judgements talking about the same propositions!

Ultimately, the answer to the initial question is: the expression $A \wedge B \supset B \wedge A$ is a proposition, part of the object language of our logic; hence $A \wedge B$ is also a proposition, not a judgement, and it would just be wrong to have a judgement where we need a proposition, which is what we would get if we wrote $A \wedge B$ *true* $\supset B \wedge A$ *true*. To use everyone's favourite expression: "it wouldn't typecheck".

A quick review

In lecture yesterday, we saw the *And* and *Implies* connectives. Recall their introduction and elimination rules:

$$\frac{A\ \mathit{true} \qquad B\ \mathit{true}}{A \wedge B\ \mathit{true}} \wedge I \qquad \frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}} \wedge E_L \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}} \wedge E_R$$

$$\begin{array}{c} \overline{A\ \mathit{true}}^{\,u} \\ \vdots \\ \dfrac{B\ \mathit{true}}{A \supset B\ \mathit{true}} \supset I^u \end{array} \qquad \frac{A \supset B\ \mathit{true} \qquad A\ \mathit{true}}{B\ \mathit{true}} \supset E$$

Why do these rules "make sense"? We'll go into a lot more (technical!) detail about this sometime next week[2], and probably on the next homework, but intuitively,

- If we want to claim that two things together are true, we have to be able to exhibit that each one is true individually, so $\wedge I$ is good.
- If we know that two things together are true, we also know that each one individually is true, so the two $\wedge E$ rules are good.

[1] We won't actually be making use of this judgement for the rest of the course; this is just an example for pedagogical purposes
[2] local soundness and completeness, if you're curious

- For the sake of introducing an implication, you are allowed to presuppose the antecedent in your proof of the consequent.

- To *use* (i.e. eliminate) an implication, you have to exhibit that the antecedent is true. Recall that this is licensed by the *substitution principle*; if we have a proof of B *true* given the assumption that A *true*, and we can separately show that A *true* holds, then we can conclude that B *true* holds with no assumptions.

- Also notice how the *numbers* of introduction/elimination rules match up. $\wedge I$ has 2 premises, so we need 2 elimination rules in order not to "lose information" by using the introduction rule.

An aside on implication and hypothetical judgements

So, what's that mysterious little ^{u} we have hovering at the side of the $\supset I$ rules?
It's actually nothing but an annotation that lets us keep track of where an *assumption* was introduced. For instance, in our presentation of the $\supset I$ rule, we label the final line with $\supset I^u$ to indicate that we've introduced an assumption that we will refer to as u above (that last word is important!).
N.B. please use different labels, e.g. u, v, if multiple assumptions are introduced in a derivation. This actually brushes up against a very deep idea in the study of logic, which you'll see in a few weeks[3]. It's also important in ensuring that both you and your grader don't get confused.

Notice that above, we said *above* was important. Although when trying to prove something we generally start from the bottom and build the proof tree working our way upwards (blasphemy! Is this even computer science?), we "read" the final proofs going from top to bottom, and when we get to a line labelled with an indication that an assumption was introduced here, the assumption is *discharged*, meaning it's no longer available for use below this line (very important!). This makes a lot of sense if you think about it — you introduce an assumption through $\supset I$ that can be used to prove the consequent of the implication, and this proof is worked out above this line. There's no reason to want, and it's entirely illegal, of course, to use this introduced assumption beyond the scope of where it was introduced.

Lastly, a pseudo-philosophical aside: when we're doing proofs, we're generally working in an empty context, one in which we have no hypotheses that we can actually work with and so all proofs we do have the caveat "assuming so-and-so". Implications are the only way we have to introduce hypotheses into our barren world, which is why anything semi-interesting will involve them.

An aside on implication being defined by disjunction

It was suggested in lecture yesterday that implication be defined by

$$\frac{\neg A \vee B\ \textit{true}}{A \supset B} \supset I$$

Reasons why we don't want to do this:

- We haven't done negation and disjunction yet!

- It's actually just wrong, because disjunction in constructive logic is different from disjunction in classical logic (spoiler).

- Generally, it's distasteful to define logical connectives in terms of other logical connectives, and it also makes things harder to do.

Protips for proofs

(These directions assume you write the thing you're trying to prove at the bottom of a sheet of paper and work upwards from there)

Generally, use introduction rules when you're working upwards on your paper, and elimination rules when you're working downwards, e.g. making use of assumptions you have.

[3]alpha equivalence, if you're curious

Examples

Self-absorbed

Question: Prove $A \supset A$ *true*.

$$\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\supset I^u$$

Maybe one is better than two

Question: Prove $A \wedge B \supset B$ *true*.

$$\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathit{true}}\,u}{B\ \mathit{true}}\wedge E_R}{A \wedge B \supset B\ \mathit{true}}\supset I^u$$

Some logical connectives too need to get to work

Question: Prove $A \wedge B \supset B \wedge A$ *true*.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathit{true}}\,u}{B\ \mathit{true}}\wedge E_R \qquad \dfrac{\dfrac{}{A \wedge B\ \mathit{true}}\,u}{A\ \mathit{true}}\wedge E_L}{B \wedge A\ \mathit{true}}\wedge I}{A \wedge B \supset B \wedge A\ \mathit{true}}\supset I^u$$

A game

Question: Prove $A \wedge (A \supset B) \supset B$ *true*.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (A \supset B)\ \mathit{true}}\,u}{A \supset B\ \mathit{true}}\wedge E_R \qquad \dfrac{\dfrac{}{A \wedge (A \supset B)\ \mathit{true}}\,u}{A\ \mathit{true}}\wedge E_L}{B\ \mathit{true}}\supset E}{A \wedge (A \supset B) \supset B\ \mathit{true}}\supset I^u$$

Just passing through

Question: Prove $A \wedge (A \supset B) \wedge (B \supset C) \supset C$ *true*.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (A \supset B) \wedge (B \supset C)\ \mathit{true}}\,u}{(A \supset B) \wedge (B \supset C)\ \mathit{true}}\wedge E_R}{B \supset C\ \mathit{true}}\wedge E_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (A \supset B) \wedge (B \supset C)\ \mathit{true}}\,u}{(A \supset B) \wedge (B \supset C)\ \mathit{true}}\wedge E_2}{A \supset B\ \mathit{true}}\wedge E_L \qquad \dfrac{\dfrac{}{A \wedge (A \supset B) \wedge (B \supset C)\ \mathit{true}}\,u}{A\ \mathit{true}}\wedge E_L}{B\ \mathit{true}}\supset E}{C\ \mathit{true}}\supset E}{A \wedge (A \supset B) \wedge (B \supset C) \supset C\ \mathit{true}}\supset I^u$$

Begone!

Question: Prove $A \supset (B \supset A)$ *true*.

$$\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u}{B \supset A\ \mathit{true}}\supset I^v}{A \supset (B \supset A)\ \mathit{true}}\supset I^u$$