Midterm 1

15-317: Constructive Logic

October 1, 2009

Name: Andrew ID:

Instructions

- This exam is closed-book, but one two-sided sheet of notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are four problems, each with several parts. Not all problems are the same size or difficulty. You have 80 minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Most important,

$\boxed{\textbf{Don't Panic}}$

Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	**Total**
Score					
Max	55	55	20	20	150
Grader					

1 Natural Deduction and Harmony (55 points)

This problem is inspired by a suggestion from a student during the first lecture on the sequent calculus. Consider the following alternative definition of conjunction:

$$\dfrac{A\ true \qquad \begin{matrix}\overline{A\ true}^{\,u} \\ \vdots \\ B\ true\end{matrix}}{A \star B\ true}\ \star I^u \qquad\qquad \dfrac{A \star B\ true}{A\ true}\ \star E_L \qquad\qquad \dfrac{A \star B\ true}{B\ true}\ \star E_R$$

The introduction rule has a new, hypothetical premise, while the elimination rules are the standard ones. We would like to show that the elimination rules are still in harmony with the new introduction rule.

Task 1 (10 pts)**.** Prove the elimination rules locally sound by giving local reductions.

(Problem continues on next page)

Task 2 (10 pts)**.** Prove the elimination rules locally complete by giving a local expansion.

Task 3 (10 pts)**.** Give rules for verifications and uses of $A \star B$.

Task 4 (10 pts)**.** Propose sequent calculus left and right rules for $A \star B$ that correspond to the introduction and elimination rules.

(Problem continues on next page)

Task 5 (5 pts)**.** Thinking of the sequent calculus as a method for performing proof search, why might we prefer this formulation of conjunction over the standard one?

Task 6 (10 pts)**.** Here is a proof term assignment for $\star I^u$:

$$\dfrac{M : A \qquad \begin{array}{c} \overline{u : A}\;u \\ \vdots \\ N : B \end{array}}{\langle M, u.\, N \rangle : A \star B}\;\star I^u$$

Propose a proof term assignment for the elimination rules and write your local reductions using only proof terms.

2 Natural Numbers and Induction (55 points)

Recall the rules for natural number arithmetic and induction (recapped in Figure 1). Consider extending arithmetic with predicates for even and odd defined by the following introduction and elimination rules.

$$\frac{}{\mathsf{even}(0)}\ \mathsf{ev}I_0 \qquad \frac{\mathsf{odd}(n)}{\mathsf{even}(\mathsf{s}\ n)}\ \mathsf{ev}I_{\mathsf{s}} \qquad \frac{\mathsf{even}(n)}{\mathsf{odd}(\mathsf{s}\ n)}\ \mathsf{od}I_{\mathsf{s}}$$

$$\frac{\mathsf{odd}(0)}{J}\ \mathsf{od}E_0 \qquad \frac{\mathsf{even}(\mathsf{s}\ n)}{\mathsf{odd}(n)}\ \mathsf{ev}E_{\mathsf{s}} \qquad \frac{\mathsf{odd}(\mathsf{s}\ n)}{\mathsf{even}(n)}\ \mathsf{od}E_{\mathsf{s}}$$

Task 1 (10 pts)**.** Show the following rule derivable:

$$\frac{\mathsf{even}(n) \lor \mathsf{odd}(n)}{\mathsf{even}(\mathsf{s}\ n) \lor \mathsf{odd}(\mathsf{s}\ n)}\ \mathsf{eo}I_{\lor}$$

Task 2 (10 pts)**.** Show the following rule derivable:

$$\frac{\mathsf{even}(\mathsf{s}\ n) \land \mathsf{odd}(\mathsf{s}\ n)}{\mathsf{even}(n) \land \mathsf{odd}(n)}\ \mathsf{eo}E_{\land}$$

(Problem continues on next page)

Task 3 (10 pts)**.** Translate the following assertions into first-order logic:

- *Every natural number is even or odd.* **(*)**

- *No natural number is both even and odd.*

Task 4 (15 pts)**.** Give a natural deduction proof of your translation of the assertion **(*)**, "*Every natural number is even or odd.*" You may use the rules you derived above.

(Problem continues on next page)

Task 5 (10 pts). We now consider the computational content of your proof. Assume we are not interested in the evidence that a number is even or odd, just in whether it is even or odd. The type of the function extracted from your proof then is

$$\mathsf{decide} : \mathsf{nat} \to 1 + 1$$

where 1 is the unit type inhabited by the unit element $\langle\,\rangle$. Give the definition of decide that corresponds to your proof. You may use the schema of primitive recursion, the primitive recursion operator R, or Tutch syntax, whichever you prefer.

3 Classical Logic (20 points)

Recall the rules for classical logic (recapped in Figure 2). In classical logic, implication may be defined in terms of negation and disjunction: $A \supset B := \neg A \vee B$.

Task 1 (15 pts)**.** Using classical natural deduction, prove $(A \supset B) \supset (\neg A \vee B)$. You may use any classical reasoning principles we've shown in lecture or in homework, including proof by contradiction (*PBC*), the law of the excluded middle (*LEM*), and double-negation elimination (*DNE*).

Task 2 (5 pts)**.** Explain informally why this theorem cannot be proven intuitionistically.

4 Mistakes Were Made (20 points)

Consider the following purported proof:

$$
\dfrac{
  \dfrac{}{\exists x{:}\tau.\, A(x)\ \mathit{true}}\,u_1
  \qquad
  \dfrac{
    \dfrac{}{\exists x{:}\tau.\, B(x)\ \mathit{true}}\,u_2
    \qquad
    \dfrac{
      \overline{a : \tau}
      \qquad
      \dfrac{
        \dfrac{}{A(a)\ \mathit{true}}\,w_1
        \qquad
        \dfrac{}{B(a)\ \mathit{true}}\,w_2
      }{A(a) \wedge B(a)\ \mathit{true}}\,{\wedge}I
    }{\exists x{:}\tau.\, A(x) \wedge B(x)\ \mathit{true}}\,\exists I
  }{\exists x{:}\tau.\, A(x) \wedge B(x)\ \mathit{true}}\,\exists E^{a,w_2}
}{
  \dfrac{
    \dfrac{\exists x{:}\tau.\, A(x) \wedge B(x)\ \mathit{true}}{(\exists x{:}\tau.\, B(x)) \supset \exists x{:}\tau.\, A(x) \wedge B(x)\ \mathit{true}}\,{\supset}I^{u_2}
  }{(\exists x{:}\tau.\, A(x)) \supset (\exists x{:}\tau.\, B(x)) \supset \exists x{:}\tau.\, A(x) \wedge B(x)\ \mathit{true}}\,{\supset}I^{u_1}
}\,\exists E^{a,w_1}
$$

Task 1 (15 pts)**.** This proof is incorrect. Circle the label(s) of the rule(s) that are applied incorrectly. Explain what is wrong with each.

Task 2 (5 pts)**.** Explain informally why the purported theorem could not possibly be true.

A Useful Rules

$$\frac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \frac{n : \mathsf{nat}}{\mathsf{s}\ n : \mathsf{nat}}\ \mathsf{nat}I_{\mathsf{s}} \qquad \frac{n : nat \quad \mathsf{C}(0)\ true \quad \begin{array}{c}\overline{x : nat}\ ,\ \overline{\mathsf{C}(x)\ true}^{\,u}\\ \vdots \\ \mathsf{C}(\mathsf{s}\ x)\ true\end{array}}{C(n)\ true}\ \mathsf{nat}E^{x,u}$$

Figure 1: Rules for natural numbers and induction.

$$A\ false := \begin{array}{c} A\ true \\ \vdots \\ \# \end{array} \qquad \frac{A\ false \quad A\ true}{J}\ contra \qquad \frac{\begin{array}{c}\overline{A\ false}^{\,k} \\ \vdots \\ \# \end{array}}{A\ true}\ PBC^k$$

$$\frac{\begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ \# \end{array}}{\neg A\ true}\ \neg I^u \qquad \frac{\neg A\ true \quad \begin{array}{c}\overline{A\ false}^{\,k} \\ \vdots \\ J \end{array}}{J}\ \neg E^k$$

Figure 2: Rules for classical natural deduction.