

Assignment 5: Sequent calculus

15-317: Constructive Logic

Due: Thursday 10/13/13 **before class**

Total points: 40

Soundness and completeness

The sequent calculus is a powerful tool for reasoning about our logics. By heavily restricting the form of proofs, the sequent calculus makes it easy to show that certain propositions (such as \perp and $\neg(A \vee \neg A)$) are *not* provable. A reasonable worry, however, is that since the sequent calculus is much more restricted than natural deduction, it may not be able to prove everything that natural deduction can. In this assignment, you will show that this is not the case. A proposition is provable in natural deduction if and only if it is provable in the sequent calculus.

A different way of writing natural deduction

In order to make reasoning about natural deduction as a system more tractable, we must adjust the notation we use for natural deduction proofs to make our assumptions more explicit. We write $A_1, \dots, A_n \vdash A$ to mean that A is provable from the assumptions A_1, \dots, A_n . Like in the sequent calculus, we will usually abbreviate the list of assumptions as Γ , and write $\Gamma \vdash A$.

$A_1, \dots, A_n \vdash A$ is equivalent to, in our original notation,

$$\frac{\overline{A_1 \text{ true}}^{u_1} \quad \dots \quad \overline{A_n \text{ true}}^{u_n}}{\vdots} \\ A \text{ true}$$

Below we show the rules for natural deduction with our new notation. Most of the rules are essentially unchanged, except for the addition of the Γ context. There is one new rule, *hyp*, which corresponds to the use of an assumption. $\vee E$ and $\supset I$ change a little, in that they now introduce assumptions by adding them to the context.

It is important to stress that this is simply a minor notational change that makes more precise what assumptions are in scope. The structure of proofs is exactly the same.

$$\frac{}{\Gamma, A \vdash A} \text{hyp} \quad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee E \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset I$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_L \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_R$$

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_L \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_R$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp E \quad \frac{}{\Gamma \vdash \top} \top I$$

In order to prove soundness of the sequent calculus, you will need to use the following theorems:

- **Weakening** For all Γ, A, C , if $\Gamma \vdash C$, then $\Gamma, A \vdash C$.
- **Substitution** For all Γ, A, C , if $\Gamma \vdash A$ and $\Gamma, A \vdash C$, then $\Gamma \vdash C$.

Sequent calculus recap

For reference, we present the rules of the sequent calculus.

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \quad \frac{\Gamma, B_1 \supset B_2 \Longrightarrow B_1 \quad \Gamma, B_1 \supset B_2, B_2 \Longrightarrow A}{\Gamma, B_1 \supset B_2 \Longrightarrow A} \supset L$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \quad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \quad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \quad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \quad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, P \Longrightarrow P} \textit{init} \quad \frac{}{\Gamma \Longrightarrow \top} \top R \quad \frac{}{\Gamma, \perp \Longrightarrow C} \perp L$$

In order to prove completeness of the sequent calculus, you will need to use the following theorems:

- **Weakening** For all Γ, A, C , if $\Gamma \Longrightarrow C$, then $\Gamma, A \Longrightarrow C$.
- **Identity** For all Γ, A : $\Gamma, A \Longrightarrow A$.
- **Cut** For all Γ, A, C , if $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$, then $\Gamma \Longrightarrow C$.

Cut is a very powerful theorem. While the theorem statement is the same as the statement of substitution in natural deduction, it is much more difficult to prove, because of how the sequent calculus breaks down assumptions in the context. We will discuss proving cut in class next week.

To make writing derivations that use these theorems easier, you may use them as if they were inference rules, but written with a dotted line. For example:

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow C}{\Gamma \Longrightarrow C} \textit{cut}$$

12 1. **Soundness**

We say that the sequent calculus is sound if every sequent provable in the sequent calculus is provable in natural deduction as well. That is to say, for all Γ, A , if $\Gamma \Longrightarrow A$, then $\Gamma \vdash A$. We prove this by induction over the derivation of $\Gamma \Longrightarrow A$.

We present an example case. Suppose that the last rule in the derivation is $\wedge L_1$. Then the derivation looks like:

$$\frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1$$

We want to show that $\Gamma, A \wedge B \vdash C$. By our induction hypothesis, we have that $\Gamma, A \wedge B, A \vdash C$. We can construct the following derivation:

$$\frac{\frac{\Gamma, A \wedge B \vdash A \wedge B}{\Gamma, A \wedge B \vdash A} \wedge E_1 \quad \frac{\Gamma, A \wedge B, A \vdash C}{\Gamma, A \wedge B \vdash C} \text{hyp}}{\Gamma, A \wedge B \vdash C} \text{subst}$$

Give the cases of the proof of soundness corresponding to the following rules:

- *init*
- $\wedge R$
- $\supset L$
- $\vee L$

12 2. **Completeness**

We say that the sequent calculus is complete if everything provable in natural deduction is provable in the sequent calculus as well. That is to say, for all Γ, A , if $\Gamma \vdash A$, then $\Gamma \Longrightarrow A$. We prove this by induction over the derivation of $\Gamma \vdash A$.

We present an example case. Suppose that the last rule in the derivation is $\supset E$. Then the derivation looks like

$$\frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset E$$

and we need to show that $\Gamma \Longrightarrow B$. By our induction hypothesis, we have that $\Gamma \Longrightarrow A \supset B$ and $\Gamma \Longrightarrow A$. We can then construct the derivation

$$\frac{\frac{\Gamma \Longrightarrow A}{\Gamma, A \supset B \Longrightarrow A} \text{weaken} \quad \frac{\Gamma, A \supset B, B \Longrightarrow B}{\Gamma, A \supset B \Longrightarrow B} \text{id}}{\Gamma \Longrightarrow B} \text{cut}$$

Give the cases of the proof of completeness corresponding to the following rules:

- *hyp*
- $\supset I$

- $\wedge E_L$
- $\vee E$

8 3. **Sequent proofs**

Give a derivation of the sequent $\Rightarrow A \wedge ((A \supset B) \vee (A \supset C)) \supset B \vee C$. If it helps tidy up your proof, you can elide parts of the context that will not be used in a branch of a proof with \dots .

8 4. **Unprovable sequents**

Show that Peirce's Law is not provable in the sequent calculus. That is, show that there does not exist a derivation of $\Rightarrow ((A \supset B) \supset A) \supset A$.