Constructive Logic (15-317), Fall 2014
Assignment 10: Linear Logic

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Due: Thursday, December 10, 2015 (before class)

In the final assignment, you will explore linear logic. First, you will do some derivations in linear logic. Then, you will have to show local soundness and completeness for parts of linear logic. Finally, you'll work through one way to interpret linear logic, and use that to prove that certain judgments are unprovable.

Your work should be submitted electronically before the beginning of the class. Please convert your homework to a PDF file titled `hw10.pdf`, and put the file in

`/afs/andrew/course/15/317/submit/<your andrew id>`

If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand and scan them.

1 Some Intuition (1 point)

Task 1 (1 point)**.** Last week, you submitted an incorrect proof. Or maybe you didn't. Either way, the logic police are coming to get you and put you in logic jail.

Fortunately, the logic police may be susceptible to bribery. If you decide to bribe them, they will tell you whether they prefer chocolates or cold, hard cash (unmarked $1 bills only, please). Then, you will hand it over, and they will decide your fate. Either they will give you a document that ensures your release, or they will do nothing and you will have to try bribing them again (if you bribe them once, you can no longer decide to go to jail for a year, since your failed bribe will result in a far harsher sentence, so you have no choice but to keep offering bribes). If you decide not to bribe them, you will spend 1 year in logic jail and then be set free.

Write a linear logic proposition that expresses the predicament from YOUR perspective.

2 A Few Final Truths (8 points)

Task 2 (8 pts)**.** For each of the following judgments, say whether they hold or not. If they hold, give a derivation in the sequent-style linear logic. If not, give a brief intuitive argument as to why (no proof is required). For two of the judgements that hold, also give a proof using the focused linear logic. But if only one or zero judgements hold, just give focused linear logic proofs for those.

1. $A \multimap (B \otimes C) \Vdash (A \multimap B) \otimes (A \multimap C)$
2. $A \multimap (B \mathbin{\&} C) \Vdash (A \multimap B) \mathbin{\&} (A \multimap C)$
3. $(A \multimap C) \oplus (B \multimap C) \Vdash (A \otimes B) \multimap C$
4. $A \multimap \mathbf{1}, (A \otimes A) \Vdash A$

3 A Harmonious End (8 points)

Task 3 (4 pts)**.** Show that the $\multimap$ connective is locally sound and complete.

Task 4 (4 pts)**.** Show that the $\&$ connective is locally sound and complete.

4 Model Theory (23 points)

It turns out that we can interpret the propositions of linear logic in terms of operations on sets of natural numbers. In this section, we will prove that this interpretation works, and use it show that certain judgments are unprovable.

Let f be some function which maps atomic propositions to sets of natural numbers. Then we can extend this to a function $[\![\cdot]\!]_f$ which maps propositions in linear logic into sets of natural numbers, as follows:

$$\begin{aligned}
[\![P]\!]_f &= f(P) \qquad \text{for } P \text{ atomic} \\
[\![X \oplus Y]\!]_f &= [\![X]\!]_f \cup [\![Y]\!]_f \\
[\![X \,\&\, Y]\!]_f &= [\![X]\!]_f \cap [\![Y]\!]_f \\
[\![X \multimap Y]\!]_f &= \{z \mid \forall x \in [\![X]\!]_f .\, x + z \in [\![Y]\!]_f\} \\
[\![X \otimes Y]\!]_f &= \{x + y \mid x \in [\![X]\!]_f, y \in [\![Y]\!]_f\} \\
[\![\mathbf{1}]\!]_f &= \{0\} \\
[\![\mathbf{0}]\!]_f &= \emptyset \\
[\![\top]\!]_f &= \mathbb{N}
\end{aligned}$$

Suppose $\Gamma = A_1, \ldots, A_n$. Write $\Gamma^{\otimes}$ for the formula $A_1 \otimes \ldots \otimes A_n$. (If $\Gamma = \cdot$, then $\Gamma^{\otimes} = \mathbf{1}$). Then, the above intepretation has the following property:

Theorem 1. *If $\Gamma \Vdash A$, then $0 \in [\![\Gamma^{\otimes} \multimap A]\!]_f$.*

Task 5 (15 pts)**.** Prove theorem 1 by induction on the derivation in linear logic. You only have to do the cases for:

- $\multimap E$
- $\otimes I$
- $\oplus E$

We can use theorem 1 as a way to prove that certain judgments are unprovable. This is an alternative to the proofs we've seen earlier in the class that used sequent calculus to make similar arguments.

Task 6 (8 pts)**.** Use theorem 1 to show that the following judgments are *not* provable, in general. In each example, P and Q are atomic propositions. (Hint: Come up with an appropriate function f and then show that 0 is not in the set we get when we apply $[\![\cdot]\!]_f$.).

- $\cdot \Vdash P \oplus (P \multimap \mathbf{0})$
- $\cdot \Vdash (P \otimes Q) \multimap (P \,\&\, Q)$