1: Operators of Differential Dynamic Logic (dL)

dL	Operator	Meaning
$e = \tilde{e}$	equals	true iff values of terms e and $\tilde{e}$ are equal
$e \geq \tilde{e}$	greater-or-equal	true iff value of e greater-or-equal to value of $\tilde{e}$
$p(e_1, \ldots, e_k)$	predicate	true iff p holds for the value of $(e_1, \ldots, e_k)$
$\neg P$	negation / not	true if P is false
$P \land Q$	conjunction / and	true if both P and Q are true
$P \lor Q$	disjunction / or	true if P is true or if Q is true
$P \to Q$	implication / implies	true if P is false or Q is true
$P \leftrightarrow Q$	bi-implication / equivalent	true if P and Q are both true or both false
$\forall x\, P$	universal quantifier / for all	true if P is true for all real values of variable x
$\exists x\, P$	existential quantifier / exists	true if P is true for some real value of variable x
$[a]P$	$[\cdot]$ modality / box	true if P is true after all runs of HP a
$\langle a \rangle P$	$\langle\cdot\rangle$ modality / diamond	true if P is true after at least one run of HP a

2: Statements and effects of Hybrid Programs (HPs)

HP	Operation	Effect
$x := e$	discrete assignment	assigns value of term e to variable x
$x := *$	nondeterministic assignment	assigns any real value to variable x
$x' = f(x) \,\&\, Q$	continuous evolution	evolve along differential equation $x' = f(x)$ within evolution domain Q for any duration
$?Q$	test	check truth of first-order formula Q at current state
$a; b$	sequential composition	HP b starts after HP a finishes
$a \cup b$	nondeterministic choice	choice between alternatives HP a or HP b
a^*	nondeterministic repetition	repeats HP a n-times for any $n \in \mathbb{N}$

3: Semantics of dL formula P in interpretation I is the set of states $[\![P]\!] \subseteq \mathcal{S}$ in which it is true

$$
\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega \in \mathcal{S} : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\} \\
[\![p(e_1, \ldots, e_k)]\!] &= \{\omega \in \mathcal{S} : (\omega[\![e_1]\!], \ldots, \omega[\![e_k]\!]) \in I(p)\} \qquad \text{for predicate symbol } p \\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathcal{S} \setminus [\![P]\!] \\
[\![P \land Q]\!] &= [\![P]\!] \cap [\![Q]\!] \\
[\![\exists x\, P]\!] &= \{\omega \in \mathcal{S} : \nu \in [\![P]\!] \text{ for some state } \nu \text{ with } \nu = \omega \text{ except for the real value of } x\} \\
[\![\forall x\, P]\!] &= \{\omega \in \mathcal{S} : \nu \in [\![P]\!] \text{ for all states } \nu \text{ with } \nu = \omega \text{ except for the real value of } x\} \\
[\![\langle a \rangle P]\!] &= [\![a]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some state } \nu \text{ such that } (\omega, \nu) \in [\![a]\!]\} \\
[\![[a]P]\!] &= [\![\neg\langle a \rangle \neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all states } \nu \text{ such that } (\omega, \nu) \in [\![a]\!]\}
\end{aligned}
$$

4: Semantics of HP a in interpretation I is relation $[\![a]\!] \subseteq \mathcal{S} \times \mathcal{S}$ between initial and final states

$$
\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) : \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) : \omega \in [\![Q]\!]\} \\
[\![x' = f(x) \,\&\, Q]\!] &= \{(\omega, \nu) : \varphi \models x' = f(x) \land Q \text{ for some solution } \varphi : [0, r] \to \mathcal{S} \text{ with } \varphi(r) = \nu, \varphi(0) = \omega|_{\{x'\}^{\complement}}\} \\
[\![a \cup b]\!] &= [\![a]\!] \cup [\![b]\!] \\
[\![a; b]\!] &= [\![a]\!] \circ [\![b]\!] = \{(\omega, \nu) : (\omega, \mu) \in [\![a]\!], (\mu, \nu) \in [\![b]\!]\} \\
[\![a^*]\!] &= [\![a]\!]^* = \bigcup_{n \in \mathbb{N}} [\![a^n]\!] \quad \text{with } a^{n+1} \equiv (a^n; a) \text{ and } a^0 \equiv (?true)
\end{aligned}
$$

[1] André Platzer. *Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics*. Springer, Heidelberg, 2010. `doi:10.1007/978-3-642-14509-4`.

[2] André Platzer. Logics of dynamical systems. In *LICS*, pages 13–24. IEEE, 2012. `doi:10.1109/LICS.2012.13`.

[3] André Platzer. A complete uniform substitution calculus for differential dynamic logic. *J. Autom. Reas.*, 2016. `doi:10.1007/s10817-016-9385-1`.

5: Axiomatization

$\langle\cdot\rangle$ $\langle a\rangle P \leftrightarrow \neg[a]\neg P$

$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$

$[:*]$ $[x := *]p(x) \leftrightarrow \forall x\, p(x)$

$[?]$ $[?Q]p \leftrightarrow (Q \to p)$

$[\cup]$ $[a \cup b]P \leftrightarrow [a]P \wedge [b]P$

$[;]$ $[a; b]P \leftrightarrow [a][b]P$

$[^*]$ $[a^*]P \leftrightarrow P \wedge [a][a^*]P$

K $[a](P \to Q) \to ([a]P \to [a]Q)$

I $[a^*]P \leftrightarrow P \wedge [a^*](P \to [a]P)$

V $p \to [a]p$

M $\dfrac{P \to Q}{[a]P \to [a]Q}$

G $\dfrac{P}{[a]P}$

$\forall$ $\dfrac{p(x)}{\forall x\, p(x)}$

MP $\dfrac{p \to q \quad p}{q}$

CT $\dfrac{f(\bar{x}) = g(\bar{x})}{c(f(\bar{x})) = c(g(\bar{x}))}$

CQ $\dfrac{f(\bar{x}) = g(\bar{x})}{p(f(\bar{x})) \leftrightarrow p(g(\bar{x}))}$

CE $\dfrac{P \leftrightarrow Q}{C(P) \leftrightarrow C(Q)}$

US $\dfrac{\varphi}{\sigma(\varphi)}$

UR $\dfrac{\varphi}{\varphi_x^y}$

BR $\dfrac{[x := e]\varphi}{[y := e]\varphi_x^y}$ $\quad (y, y', x' \notin \mathrm{FV}(\varphi))$

6: Differential equation axioms and differential axioms

DW $[x' = f(x)\,\&\, Q]P \leftrightarrow [x' = f(x)\,\&\, Q](Q \to P)$

DC $\big([x' = f(x)\,\&\, Q]P \leftrightarrow [x' = f(x)\,\&\, Q \wedge C]P\big) \leftarrow [x' = f(x)\,\&\, Q]C$

DE $[x' = f(x)\,\&\, Q]P \leftrightarrow [x' = f(x)\,\&\, Q][x' := f(x)]P$

DI $\big([x' = f(x)\,\&\, Q]P \leftrightarrow [?Q]P\big) \leftarrow (Q \to [x' = f(x)\,\&\, Q](P)')$

DG $[x' = f(x)\,\&\, Q]P \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x)\,\&\, Q]P$

DS $[x' = c\,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(x + cs)) \to [x := x + ct]p(x)\big)$

c' $(c)' = 0$

x' $(x)' = x'$

$+'$ $(e + d)' = (e)' + (d)'$

$\cdot'$ $(e \cdot d)' = (e)' \cdot d + e \cdot (d)'$

$\circ'$ $[y := g(x)][y' := 1]\big((f(g(x)))' = (f(y))' \cdot (g(x))'\big)$

7: First-order axioms

$\forall$i $(\forall x\, p(x)) \to p(c)$

$\forall\!\to$ $\forall x\, (P \to Q) \to (\forall x\, P \to \forall x\, Q)$

$V_\forall$ $p \to \forall x\, p$

8: dL Sequent calculus proof rules

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [a]J \quad J \vdash P}{\Gamma \vdash [a^*]P, \Delta}$$

$$\text{MR}\ \frac{\Gamma \vdash [a]Q, \Delta \quad Q \vdash P}{\Gamma \vdash [a]P, \Delta}$$

$$\text{ML}\ \frac{\Gamma, [a]Q \vdash \Delta \quad P \vdash Q}{\Gamma, [a]P \vdash \Delta}$$

$$\text{G}\ \frac{P}{\Gamma \vdash [a]P, \Delta}$$

$$\text{GVR}\ \frac{\Gamma_{\text{const}} \vdash p, \Delta_{\text{const}}}{\Gamma \vdash [a]p, \Delta}$$

$$\text{CTR}\ \frac{f(\bar{x}) = g(\bar{x})}{\Gamma \vdash c(f(\bar{x})) = c(g(\bar{x})), \Delta}$$

$$\text{CTL}\ \frac{f(\bar{x}) = g(\bar{x})}{\Gamma, c(f(\bar{x})) = c(g(\bar{x})) \vdash \Delta}$$

$$\text{CQR}\ \frac{\Gamma \vdash p(g(\bar{x})), \Delta \quad f(\bar{x}) = g(\bar{x})}{\Gamma \vdash p(f(\bar{x})), \Delta}$$

$$\text{CQL}\ \frac{\Gamma, p(g(\bar{x})) \vdash \Delta \quad f(\bar{x}) = g(\bar{x})}{\Gamma, p(f(\bar{x})) \vdash \Delta}$$

$$\text{CER}\ \frac{\Gamma \vdash C(Q), \Delta \quad P \leftrightarrow Q}{\Gamma \vdash C(P), \Delta}$$

$$\text{CEL}\ \frac{\Gamma, C(Q) \vdash \Delta \quad P \leftrightarrow Q}{\Gamma, C(P) \vdash \Delta}$$

$$\text{US}\ \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \sigma(\Delta)}$$

$$\text{UR}\ \frac{\Gamma\frac{y}{x} \vdash \Delta\frac{y}{x}}{\Gamma \vdash \Delta}$$

$$\text{BRR}\ \frac{\Gamma \vdash [y := e]\varphi\frac{y}{x}, \Delta}{\Gamma \vdash [x := e]\varphi, \Delta}$$

$$\text{BRL}\ \frac{\Gamma, [y := e]\varphi\frac{y}{x} \vdash \Delta}{\Gamma, [x := e]\varphi \vdash \Delta}$$

$y, y', x' \notin \text{FV}(\varphi)$

9: Differential equation sequent calculus proof rules

$$\text{dW}\ \frac{\Gamma_{\text{const}}, q(x) \vdash p(x), \Delta_{\text{const}}}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]p(x), \Delta}$$

$$\text{dI}\ \frac{\Gamma, q(x) \vdash p(x), \Delta \quad q(x) \vdash [x' := f(x)](p(x))'}{\Gamma \vdash [x' = f(x) \,\&\, q(x)]p(x), \Delta}$$

$$\text{dC}\ \frac{\Gamma \vdash [x' = f(x) \,\&\, Q]C, \Delta \quad \Gamma \vdash [x' = f(x) \,\&\, Q \wedge C]P, \Delta}{\Gamma \vdash [x' = f(x) \,\&\, Q]P, \Delta}$$

10: Propositional sequent calculus proof rules

$$\neg\text{R}\ \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \neg P, \Delta} \qquad \wedge\text{R}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma \vdash Q, \Delta}{\Gamma \vdash P \wedge Q, \Delta} \qquad \vee\text{R}\ \frac{\Gamma \vdash P, Q, \Delta}{\Gamma \vdash P \vee Q, \Delta} \qquad \rightarrow\text{R}\ \frac{\Gamma, P \vdash Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

$$\neg\text{L}\ \frac{\Gamma \vdash P, \Delta}{\Gamma, \neg P \vdash \Delta} \qquad \wedge\text{L}\ \frac{\Gamma, P, Q \vdash \Delta}{\Gamma, P \wedge Q \vdash \Delta} \qquad \vee\text{L}\ \frac{\Gamma, P \vdash \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \vee Q \vdash \Delta} \qquad \rightarrow\text{L}\ \frac{\Gamma \vdash P, \Delta \quad \Gamma, Q \vdash \Delta}{\Gamma, P \rightarrow Q \vdash \Delta}$$

$$\text{id}\ \frac{}{\Gamma, P \vdash P, \Delta} \qquad \top\text{R}\ \frac{}{\Gamma \vdash \mathit{true}, \Delta} \qquad \text{WR}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash P, \Delta} \qquad \text{PR}\ \frac{\Gamma \vdash Q, P, \Delta}{\Gamma \vdash P, Q, \Delta}$$

$$\text{cut}\ \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \bot\text{L}\ \frac{}{\Gamma, \mathit{false} \vdash \Delta} \qquad \text{WL}\ \frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \text{PL}\ \frac{\Gamma, Q, P \vdash \Delta}{\Gamma, P, Q \vdash \Delta}$$

11: Quantifier sequent calculus proof rules

$$\forall\text{R}\ \frac{\Gamma \vdash p(y), \Delta}{\Gamma \vdash \forall x\, p(x), \Delta} \quad (y \notin \Gamma, \Delta, \forall x\, p(x)) \qquad \exists\text{R}\ \frac{\Gamma \vdash p(e), \Delta}{\Gamma \vdash \exists x\, p(x), \Delta}$$

$$\forall\text{L}\ \frac{\Gamma, p(e) \vdash \Delta}{\Gamma, \forall x\, p(x) \vdash \Delta} \qquad \exists\text{L}\ \frac{\Gamma, p(y) \vdash \Delta}{\Gamma, \exists x\, p(x) \vdash \Delta} \quad (y \notin \Gamma, \Delta, \exists x\, p(x))$$

12: Derived rules

$$W\text{R}\ \frac{\vdash P}{\Gamma \vdash P, \Delta} \qquad \text{cutR}\ \frac{\Gamma \vdash Q, \Delta \quad \Gamma \vdash Q \rightarrow P, \Delta}{\Gamma \vdash P, \Delta} \qquad \leftrightarrow c\text{R}\ \frac{\Gamma \vdash Q \leftrightarrow P, \Delta}{\Gamma \vdash P \leftrightarrow Q, \Delta}$$

$$W\text{L}\ \frac{P \vdash}{\Gamma, P \vdash \Delta} \qquad \text{cutL}\ \frac{\Gamma, Q \vdash \Delta \quad \Gamma \vdash P \rightarrow Q, \Delta}{\Gamma, P \vdash \Delta} \qquad \leftrightarrow c\text{L}\ \frac{\Gamma, Q \leftrightarrow P \vdash \Delta}{\Gamma, P \leftrightarrow Q \vdash \Delta}$$

$$W\text{LR}\ \frac{P \vdash Q}{\Gamma, P \vdash Q, \Delta} \qquad \rightarrow 2 \leftrightarrow\ \frac{\Gamma \vdash P \leftrightarrow Q, \Delta}{\Gamma \vdash P \rightarrow Q, \Delta}$$

13: Derived axioms

$[]\wedge\ \boldsymbol{[\alpha](P \wedge Q)} \leftrightarrow [\alpha]P \wedge [\alpha]Q$