15: Winning Strategies & Regions
Logical Foundations of Cyber-Physical Systems

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Outline

1. Learning Objectives
2. Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics
3. Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly
4. Summary
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   • Differential Game Logic Semantics
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4 Summary
Learning Objectives

Winning Strategies & Regions

- fundamental principles of computational thinking
- logical extensions
- PL modularity principles
- compositional extensions
- differential game logic
- denotational vs. operational semantics

- adversarial dynamics
- adversarial semantics
- adversarial repetitions
- fixpoints

CT
M&C
CPS

- CPS semantics
- multi-agent operational-effects
- mutual reactions
- complementary hybrid systems
**Definition (Hybrid game $\alpha$)**

\[
\alpha, \beta ::= x := e \mid \alpha \cup \beta \mid \alpha; \beta \mid x' = f(x) \& Q \mid \alpha^* \mid \alpha^d
\]

**Definition (dGL Formula $P$)**

\[
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \& Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha]P
\]
### Definition (Hybrid game $\alpha$)

$$\alpha, \beta ::= \ x := e \ | \ ?Q \ | \ x' = f(x) \ & \ Q \ | \ \alpha \cup \beta \ | \ \alpha; \beta \ | \ \alpha^* \ | \ \alpha^d$$

### Definition (dGL Formula $P$)

$$P, Q ::= \ e \geq \tilde{e} \ | \ \neg P \ | \ P \ & \ Q \ | \ \forall x \ P \ | \ \exists x \ P \ | \ \langle \alpha \rangle P \ | \ [\alpha]P$$
### Differential Game Logic: Syntax

#### Definition (Hybrid game $\alpha$)

$$
\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) & Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d
$$

#### Definition (dGL Formula $P$)

$$
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x\; P \mid \exists x\; P \mid \langle \alpha \rangle P \mid [\alpha] P
$$
Differential Game Logic: Syntax

**Definition (Hybrid game $\alpha$)**

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

**Definition (dGL Formula $P$)**

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x P \mid \exists x P \mid \langle \alpha \rangle P \mid [\alpha] P$$
Definition (Hybrid game $\alpha$)

$$\begin{align*}
\alpha, \beta & ::= \ x := e \ | \ ?Q \ | \ x' = f(x) \ & Q \ | \ \alpha \cup \beta \ | \ \alpha; \beta \ | \ \alpha^* \ | \ \alpha^d
\end{align*}$$

Definition (dGL Formula $P$)

$$\begin{align*}
P, Q & ::= \ e \geq \bar{e} \ | \ \neg P \ | \ P \land Q \ | \ \forall x \ P \ | \ \exists x \ P \ | \ \langle \alpha \rangle P \ | \ [\alpha] P
\end{align*}$$
Differential Game Logic: Syntax

Definition (Hybrid game $\alpha$)

$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$

Definition (dGL Formula $P$)

$P, Q ::= e \geq \bar{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha] P$

“Angel has Wings $\langle \alpha \rangle$”

All Reals, Some Reals, Angel Wins, Demon Wins
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4. Summary

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Definition (dGL Formula \( P \)) \([\cdot]\) : \( \text{Fml} \rightarrow \wp(S) \)

\[
\begin{align*}
[e_1 \geq e_2] &= \{\omega \in S : \omega[e_1] \geq \omega[e_2]\} \\
[\neg P] &= ([P])^c \\
[P \land Q] &= [P] \cap [Q] \\
\langle \alpha \rangle P &= \varsigma_\alpha([P]) \quad \{\omega: \nu \in [P] \text{ for some } \nu \text{ with } (\omega, \nu) \in [\alpha]\} \\
[[\alpha]P] &= \delta_\alpha([P])
\end{align*}
\]
Differential Game Logic: Denotational Semantics

Definition (Hybrid game $\alpha$)

\[
[\cdot] : \text{Fml} \rightarrow \wp(\mathcal{S})
\]

\begin{align*}
[e_1 \geq e_2] &= \{\omega \in \mathcal{S} : \omega[e_1] \geq \omega[e_2]\} \\
[-P] &= ([P])^C \\
[P \land Q] &= [P] \cap [Q] \\
\langle \alpha \rangle P &= \varsigma_{\alpha}([P]) \quad \{\omega : \nu \in [P] \text{ for some } \nu \text{ with } (\omega, \nu) \in [\alpha]\} \\
[[\alpha]P] &= \delta_{\alpha}([P])
\end{align*}

Only for HPs. No interactive play!
Definition (Hybrid game $\alpha$: denotational semantics)

\[
\mathcal{S}_x := e(X) =
\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_x := e(X) = \{ \omega \in S : \omega^x_e \in X \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\exists x' = f(x) \& Q(X) = \{ \phi(0) \in S: \phi(r) \in X \text{ for an } r \text{ and } \phi |_{=} x' = f(x) \land Q(X) \}$$
\[ \varsigma x' = f(x) \land Q(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for an } r \text{ and } \varphi \models x' = f(x) \land Q \} \]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\mathcal{S}_Q(X) = \mathcal{S}_Q(X) \cap X$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s?Q(X) = \llbracket Q \rrbracket \cap X$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\varsigma_{\alpha \cup \beta}(X) =$

$\varsigma_{\alpha}(X) \cup \varsigma_{\beta}(X)$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{\alpha \cup \beta}(X) = s_\alpha(X) \cup s_\beta(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\varsigma_{\alpha;\beta}(X) =$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{\alpha;\beta}(X) = s_{\alpha}(s_{\beta}(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\varsigma_{\alpha d}(X) =$

![Diagram of $X$]
Definition (Hybrid game \( \alpha \): denotational semantics)

\[
\varsigma_{\alpha^d}(X) =
\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\mathcal{S}_{\alpha^d}(X) =$$

\[ X^c \subseteq X^c \subseteq \mathcal{S}_\alpha(X^c)^c \subseteq \mathcal{S}_\alpha(X^c) \]

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Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha^d}(X) = (\varsigma_{\alpha}(X^c))^c$$

[Diagram: A diagram illustrating the relationship between $X$, $X^c$, $s_{\alpha^d}(X)$, and $s_{\alpha}(X^c)^c$.]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x:=e}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$
\delta_{x:=e}(X) = \{ \omega \in S : \omega_x^{[e]} \in X \}
$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x'} = f(x) \& Q(X) = \{\phi(0) \in S : \phi(r) \in X \text{ for all } r \text{ with } \phi|_r = x' = f(x) \& Q(X)\}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$\delta_{x'=f(x)} \& Q(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for all } r \text{ with } \varphi \models x' = f(x) \land Q \}$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_Q(X) = [Q] \cap X$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta?Q(X) = \lceil Q \rceil \cup X$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha \cup \beta}(X) =$$

\[\delta_{\alpha}(X) \cap \delta_{\beta}(X)\]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha \cup \beta}(X) = \delta_{\alpha}(X) \cap \delta_{\beta}(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$
\delta_{\alpha;\beta}(X) = 
$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha;\beta}(X) = \delta_{\alpha}(\delta_{\beta}(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$
\delta_{\alpha_d}(X) = \delta_{\alpha_d}(X^c) \cup \delta_{\alpha_d}(X^c)
$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$
\delta_{\alpha^d}(X) = (\delta_{\alpha}(X^C))^C
$$
### Differential Game Logic: Denotational Semantics

#### Definition (Hybrid game $\alpha$)

\begin{align*}
ς_x := & e(X) = \{ \omega \in S : \omega_x^e \in X \} \\
ς_{x'} := & f(x)(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \} \\
ς?Q(X) := & [Q] \cap X \\
ς_\alpha \cup \beta(X) := & ς_\alpha(X) \cup ς_\beta(X) \\
ς_\alpha ; \beta(X) := & ς_\alpha(ς_\beta(X)) \\
ς_\alpha^*(X) := & \quad \\
ς_\alpha^d(X) := & (ς_\alpha(X^C))^C
\end{align*}

#### Definition (dGL Formula $P$)

\begin{align*}
[e_1 \geq e_2] := & \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \} \\
[\neg P] := & (\llbracket P \rrbracket)^C \\
[P \land Q] := & \llbracket P \rrbracket \cap \llbracket Q \rrbracket \\
[\langle \alpha \rangle P] := & ς_\alpha(\llbracket P \rrbracket) \\
[\lbrack \alpha \rbrack P] := & \delta_\alpha(\llbracket P \rrbracket)
\end{align*}
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4 Summary
Filibusters & The Significance of Finitude

\[(x := 0 \cap x := 1)^* \] x = 0

wfd \[\rightsquigarrow\] false unless \( x = 0 \)
Definition (Hybrid game $\alpha$)

$\mathcal{S}_{\alpha^*}(X) =$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

$$[\alpha^*] = \bigcup_{n \in \mathbb{N}} [\alpha^n]$$

where $\alpha^{n+1} \equiv \alpha^n$; $\alpha^0 \equiv \text{true}$ for HP $\alpha$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle x \neq 1$$
Semantics of Repetition Advance Notice Semantics

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

advance notice semantics?

$$x = 1 \land a = 1 \rightarrow ((x := a; a := 0) \cap x := 0)^* x \neq 1$$
**Definition (Hybrid game $\alpha$)**

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

too hard to predict all iterations!

$$x = 1 \land a = 1 \rightarrow \langle ((x := a; a := 0) \cap x := 0)^* \rangle x \neq 1$$
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

Since \( s_\alpha(Y) \) is just one round away from \( Y \).
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s^*_\alpha(X) = \bigcup_{n \in \mathbb{N}} s^n_\alpha(X)$$

$$s^0_\alpha(X) \overset{\text{def}}{=} X$$

$$s^{\kappa+1}_\alpha(X) \overset{\text{def}}{=} X \cup s_\alpha(s^\kappa_\alpha(X))$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s^*_\alpha(X) = \bigcup_{n \in \mathbb{N}} s^n_\alpha(X)$$

$$s^0_\alpha(X) \overset{\text{def}}{=} X$$

$$s^{\kappa+1}_\alpha(X) \overset{\text{def}}{=} X \cup s_\alpha(s^\kappa_\alpha(X))$$
\[ \omega \text{-Semantics} \]

**Definition (Hybrid game } \alpha \text{)**

\[ s_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} s_\alpha^n(X) \]

\[ s_\alpha^0(X) \overset{\text{def}}{=} X \]

\[ s_\alpha^\kappa+1(X) \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^\kappa(X)) \]

\[ s_\alpha^2(X) \subseteq s_\alpha(X) \subseteq X \]
Definition (Hybrid game $\alpha$)

$$s^*_\alpha(X) = \bigcup_{n \in \mathbb{N}} s^n_\alpha(X)$$

$n$ outside the game so Demon won't know

$$s^0_\alpha(X) \overset{\text{def}}{=} X$$

$$s^{\kappa+1}_\alpha(X) \overset{\text{def}}{=} X \cup s_\alpha(s^{\kappa}_\alpha(X))$$
Definition (Hybrid game $\alpha$)

$$s_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} s_\alpha^n(X)$$

$$s_\alpha^0(X) \overset{\text{def}}{=} X$$

$$s_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^\kappa(X))$$

Example

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle \ (0 \leq x < 1)$$
Semantics of Repetition

**ω-Semantics**

**Definition (Hybrid game \( \alpha \))**

\[
\varsigma_{\alpha^*(X)} = \bigcup_{n \in \mathbb{N}} \varsigma_{\alpha}^n(X)
\]

\[
\varsigma_{\alpha}^0(X) \overset{\text{def}}{=} X
\]

\[
\varsigma_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_{\alpha}(\varsigma_{\alpha}^{\kappa}(X))
\]

**Example**

\[
\{ (x := 1; x' = 1^d \cup x := x - 1)^* \} (0 \leq x < 1) \quad \varsigma_{\alpha}^n([0, 1)) = [0, n) \not\in \mathbb{R}
\]
Semantics of Repetition

**ω-Semantics**

**Definition (Hybrid game \( \alpha \))**

\[
\varsigma_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n(X)
\]

\[
\begin{align*}
\varsigma_\alpha^0(X) & \overset{\text{def}}{=} X \\
\varsigma_\alpha^{\kappa+1}(X) & \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X)) \\
\varsigma_\alpha^\lambda(X) & \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} \varsigma_\alpha^\kappa(X) & \lambda \neq 0 \text{ a limit ordinal}
\end{align*}
\]

**Example**

\[
\langle (x := 1; x' = 1d \cup x := x - 1)^* \rangle (0 \leq x < 1)
\]

\[
\begin{align*}
\varsigma_\alpha^n([0, 1)) &= [0, n) \neq \mathbb{R} \\
\varsigma_\alpha^\omega([0, 1)) &= \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n([0, 1)) = [0, \infty) \neq \mathbb{R}
\end{align*}
\]
### Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$s_{\alpha}^*(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha}^n(X)$$

- $s_{\alpha}^0(X) \overset{\text{def}}{=} X$
- $s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^{\kappa}(X))$
- $s_{\alpha}^{\lambda}(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s_{\alpha}^{\kappa}(X)$  \hspace{1cm} $\lambda \neq 0$ a limit ordinal

### Example

- $\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$
- $s_{\alpha}^{\omega+1}([0, 1)) = s_{\alpha}([0, \infty)) = \mathbb{R}$
- $s_{\alpha}^{\omega}([0, 1)) = \bigcup_{n \in \mathbb{N}} s_{\alpha}^n([0, 1)) = [0, \infty) \neq \mathbb{R}$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n(X)$$

$$\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$$

$$\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$$

$$\varsigma_\alpha^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} \varsigma_\alpha^\kappa(X) \quad \lambda \neq 0 \text{ a limit ordinal}$$

$\lambda \neq 0$ a limit ordinal
Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \varsigma_\alpha^n(X)$$

- $\varsigma_\alpha^0(X) \overset{\text{def}}{=} X$
- $\varsigma_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^\kappa(X))$
- $\varsigma_\alpha^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} \varsigma_\alpha^\kappa(X)$ \quad $\lambda \neq 0$ a limit ordinal
Theorem

*Hybrid game closure ordinal* $> \omega^\omega$
Expedition: Ordinal Arithmetic

\[ \iota + 0 = \iota \]
\[ \iota + (\kappa + 1) = (\iota + \kappa) + 1 \quad \text{successor } \kappa + 1 \]
\[ \iota + \lambda = \bigcup_{\kappa < \lambda} (\iota + \kappa) \quad \text{limit } \lambda \]
\[ \iota \cdot 0 = 0 \]
\[ \iota \cdot (\kappa + 1) = (\iota \cdot \kappa) + \iota \quad \text{successor } \kappa + 1 \]
\[ \iota \cdot \lambda = \bigcup_{\kappa < \lambda} (\iota \cdot \kappa) \quad \text{limit } \lambda \]
\[ \iota^0 = 1 \]
\[ \iota^{\kappa + 1} = \iota^\kappa \cdot \iota \quad \text{successor } \kappa + 1 \]
\[ \iota^{\lambda} = \bigcup_{\kappa < \lambda} \iota^\kappa \quad \text{limit } \lambda \]
\[ 2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4 \]
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^\kappa(X)$$

$$s_{\alpha}^0(X) \overset{\text{def}}{=} X$$

$$s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^\kappa(X))$$

$$s_{\alpha}^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s_{\alpha}^\kappa(X) \quad \lambda \neq 0 \text{ a limit ordinal}$$
Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa<\infty} \varsigma_{\alpha}^{\kappa}(X)$$
Semantics of Repetition

Inflationary Semantics

Definition (Hybrid game $\alpha$)

$$s_{\alpha}^*(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X)$$
Definition (Hybrid game $\alpha$)

$$s_{\alpha}^*(X) = \bigcup_{\kappa < \infty} s_{\alpha}^\kappa(X)$$
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X)$$
Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{\kappa < \infty} \varsigma_\alpha^\kappa(X)$$
Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa < \infty} \varsigma_{\alpha}^{\kappa}(X)$$

requires transfinite patience
Implicit Definitions

The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell
Note (+1 argument)

\[ Y \subseteq s_{\alpha^*}(X) \text{ then } s_{\alpha}(Y) \subseteq s_{\alpha^*}(X) \]

Since \( s_{\alpha}(Y) \) is just one round away from \( Y \).
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} \overset{\text{def}}{s_\alpha^*(X)} \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
Note (+1 argument)

\[ Y \subseteq \mathcal{s}_\alpha^*(X) \text{ then } \mathcal{s}_\alpha(Y) \subseteq \mathcal{s}_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} \mathcal{s}_\alpha^*(X) \text{ then } \mathcal{s}_\alpha(Z) \subseteq \mathcal{s}_\alpha^*(X) = Z \]

- Which \( Z \) with \( \mathcal{s}_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( \mathcal{s}_\alpha^!Q \emptyset = \mathcal{s}_\alphaQ(\emptyset^C)^C = ([Q] \cap S)^C = [Q]^C \not\subseteq \emptyset \)
Note (+1 argument)

\[ Y \subseteq \mathcal{s}_\alpha^*(X) \text{ then } \mathcal{s}_\alpha(Y) \subseteq \mathcal{s}_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} \mathcal{s}_\alpha^*(X) \text{ then } \mathcal{s}_\alpha(Z) \subseteq \mathcal{s}_\alpha^*(X) = Z \]

- Which \( Z \) with \( \mathcal{s}_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( \mathcal{s}_\square Q_d(\emptyset) = \mathcal{s}_\square Q(\emptyset^c)^c = (\llbracket Q \rrbracket \cap S)^c = \llbracket Q \rrbracket^c \nsubseteq \emptyset \)
- Then: \( \mathcal{s}_\square Q_d([\neg Q]) = \mathcal{s}_\square Q([\neg Q]^c)^c = (\llbracket Q \rrbracket \cap [Q])^c = [\neg Q] \subseteq [\neg Q] \)
Note (+1 argument)

\[ Y \subseteq \alpha^*(X) \text{ then } \alpha(Y) \subseteq \alpha^*(X) \]

\[ Z \overset{\text{def}}{=} \alpha^*(X) \text{ then } \alpha(Z) \subseteq \alpha^*(X) = Z \]

- Which \( Z \) with \( \alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( \alpha_Q(\emptyset) = \alpha_Q(\emptyset^C)^C = ([Q] \cap S)^C = [Q]^C \not\subseteq \emptyset \)
- Then: \( \alpha_{Q^d}(\neg Q) = \alpha_Q(\neg Q)^C = ([Q] \cap [Q])^C = [\neg Q] \subseteq [\neg Q] \)
- Still too small: \( X \subseteq Z \) since Angel may decide not to repeat
Fixpoints and Pre-Fixpoints

**Definition (Pre-fixpoint)**

\[ X \cup \mathcal{S}_\alpha(Z) \subseteq Z \]

for the winning region \( Z \overset{\text{def}}{=} \mathcal{S}_{\alpha^*}(X) \)

Where is the right one?
Are there multiple such \( Z \)? Does such a \( Z \) exist?

Existence: \( Z = S \) but that's too big and independent of \( \alpha \)

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LFCPS/15: Winning Strategies & Regions
Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

\[ X \cup \mathcal{S}_\alpha(Z) \subseteq Z \]

for the winning region \( Z \overset{\text{def}}{=} \mathcal{S}_{\alpha^*}(X) \)

- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
Fixpoints and Pre-Fixpoints

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- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
- Existence: \( Z = S \)
Fixpoints and Pre-Fixpoints

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for the winning region \( Z \overset{\text{def}}{=} \mathcal{S}_{\alpha^*}(X) \)

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- Existence: \( Z = S \) but that’s too big and independent of \( \alpha \)
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

\[ X \cup \varsigma_\alpha(Y) \subseteq Y \]
\[ X \cup \varsigma_\alpha(Z) \subseteq Z \]

are pre-fixpoints, then

\[ Y \cap Z \]

is a smaller pre-fixpoint.

Proof. 
\[ X \cup \varsigma_\alpha(Y) \subseteq \]
\[ X \cup \varsigma_\alpha(Y) \cap \varsigma_\alpha(Z) \]

above \[ \subseteq Y \cap Z \]

Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!

So: repetition semantics is the smallest pre-fixpoint (well-founded)

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### Lemma (Intersection closure)

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\]

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### Proof.

\[
X \cup \varsigma_\alpha(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\varsigma_\alpha(Y) \cap \varsigma_\alpha(Z)) \overset{\text{above}}{\subseteq} Y \cap Z
\]
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Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcap\{Z \subseteq S : X \cup s_{\alpha}(Z) \subseteq Z\}$$

$X \cup s_{\alpha}(s_{\alpha^*}(X)) \subseteq s_{\alpha^*}(X)$

$s_{\alpha^*}(X)$ intersection of solutions
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcap\{Z \subseteq S : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$

$Z \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha(Z) \subseteq \varsigma_\alpha(\varsigma_\alpha^*(X))$

$\varsigma_\alpha^*(X)$ intersection of solutions by mon since $Z \subseteq \varsigma_\alpha^*(X)$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s^*_\alpha(X) = \bigcap \{Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z\}$$

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$s^*_\alpha(X)$ intersection of solutions

by mon since $Z \subseteq s^*_\alpha(X)$
Semantics of Repetition

**Definition (Hybrid game \( \alpha \))**

\[
\varsigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \varsigma_\alpha(Z) \subseteq Z \}
\]

\[
\varsigma_\alpha^*(X) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)
\]

\(
Z \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)
\)

\[
X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z
\]

\[
\varsigma_\alpha^*(X) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z
\]

\(\varsigma_\alpha^*(X)\) intersection of solutions by mon since \(Z \subseteq \varsigma_\alpha^*(X)\)

\(\varsigma_\alpha^*(X)\) smallest such \(Z\)
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\mathcal{S}_\alpha^*(X) = \bigcap \{Z \subseteq S : X \cup \mathcal{S}_\alpha(Z) \subseteq Z\}$$

$\mathcal{S}_\alpha^*(X)$ intersection of solutions

$X \cup \mathcal{S}_\alpha(Z) \subseteq X \cup \mathcal{S}_\alpha(\mathcal{S}_\alpha^*(X)) = Z$

by mon since $Z \subseteq \mathcal{S}_\alpha^*(X)$

$\mathcal{S}_\alpha^*(X) \subseteq X \cup \mathcal{S}_\alpha(\mathcal{S}_\alpha^*(X)) = Z$

since $\mathcal{S}_\alpha^*(X)$ smallest such $Z$

$Z \overset{\text{def}}{=} X \cup \mathcal{S}_\alpha(\mathcal{S}_\alpha^*(X)) \subseteq \mathcal{S}_\alpha^*(X)$
Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$s_\alpha^*(X) = \bigcap \{Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z\}$$

\[\begin{align*}
Z & \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^*(X)) \subseteq s_\alpha^*(X) \\
X \cup s_\alpha(Z) & \subseteq X \cup s_\alpha(s_\alpha^*(X)) = Z \\
s_\alpha^*(X) & = X \cup s_\alpha(s_\alpha^*(X)) = Z
\end{align*}\]

$s_\alpha^*(X)$ intersection of solutions by mon since $Z \subseteq s_\alpha^*(X)$

since $s_\alpha^*(X)$ smallest such $Z$
Definition (Hybrid game $\alpha$)

$\varsigma^*(X) = \bigcap\{Z \subseteq S : X \cup \varsigma^\alpha(Z) = Z\}$

$Z \overset{\text{def}}{=} X \cup \varsigma^\alpha(\varsigma^*(X)) \subseteq \varsigma^*(X)$

$X \cup \varsigma^\alpha(Z) \subseteq X \cup \varsigma^\alpha(\varsigma^*(X)) = Z$ by mon since $Z \subseteq \varsigma^*(X)$

$\varsigma^*(X) = X \cup \varsigma^\alpha(\varsigma^*(X)) = Z$ since $\varsigma^*(X)$ smallest such $Z$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha}(X) = \bigcap\{Z \subseteq S : X \cup s_{\alpha}(Z) = Z\} = \bigcup_{\kappa < \infty} s_{\alpha}^\kappa(X) \text{ by Knaster-Tarski}$$

$Z \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}(X)) \subseteq s_{\alpha}(X)$

$X \cup s_{\alpha}(Z) \subseteq X \cup s_{\alpha}(s_{\alpha}(X)) = Z \text{ by mon since } Z \subseteq s_{\alpha}(X)$

$s_{\alpha}(X) = X \cup s_{\alpha}(s_{\alpha}(X)) = Z \text{ since } s_{\alpha}(X) \text{ smallest such } Z$
Outline

1 Learning Objectives

2 Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics

3 Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly

4 Summary
### Definition (Hybrid game \( \alpha \))

\[
\begin{align*}
\varsigma_x := e(X) &= \{ \omega \in S : \omega_{\mathcal{X}}[e] \in X \} \\
\varsigma_x' = f(x)(X) &= \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \} \\
\varsigma?Q(X) &= [\! [Q] \!] \cap X \\
\varsigma\alpha \cup \varsigma\beta(X) &= \varsigma\alpha(X) \cup \varsigma\beta(X) \\
\varsigma\alpha ; \beta(X) &= \varsigma\alpha(\varsigma\beta(X)) \\
\varsigma\alpha^*(X) &= \bigcup_{\kappa < \infty} \varsigma\kappa^\alpha(X) \\
\varsigma\alpha^d(X) &= (\varsigma\alpha(X^c))^c
\end{align*}
\]

### Definition (dGL Formula \( P \))

\[
\begin{align*}
[\! [e_1 \geq e_2] \!] &= \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \} \\
[\! [\neg P] \!] &= (\![ P ]\!^c \\
[\! [P \land Q] \!] &= [\! [P] \!] \cap [\! [Q] \!] \\
[\! [\langle \alpha \rangle P] \!] &= \varsigma\alpha([\! [P] \!]) \\
[\! [[\alpha] P] \!] &= \delta\alpha([\! [P] \!])
\end{align*}
\]
### Differential Game Logic: Denotational Semantics

#### Definition (Hybrid game $\alpha$)

\( [\cdot] : \text{HG} \rightarrow (\wp(S) \rightarrow \wp(S)) \)

- \( \varsigma_x := e(X) = \{ \omega \in S : \omega^x[e] \in X \} \)
- \( \varsigma_{x'} = f(x)(X) = \{ \varphi(0) \in S : \varphi(r) \in X \text{ for some } r \geq 0 \text{ and } \varphi \models x' = f(x) \} \)
- \( \varsigma ? Q(X) = [Q] \cap X \)
- \( \varsigma_{\alpha \cup \beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X) \)
- \( \varsigma_{\alpha ; \beta}(X) = \varsigma_\alpha(\varsigma_\beta(X)) \)
- \( \varsigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \varsigma_\alpha(Z) \subseteq Z \} \)
- \( \varsigma_\alpha^d(X) = (\varsigma_\alpha(X^C))^C \)

#### Definition (dGL Formula $P$)

\( [\cdot] : \text{Fml} \rightarrow \wp(S) \)

- \( [e_1 \geq e_2] = \{ \omega \in S : \omega^e[e_1] \geq \omega^e[e_2] \} \)
- \( [\neg P] = ([P])^C \)
- \( [P \land Q] = [P] \cap [Q] \)
- \( [\langle \alpha \rangle P] = \varsigma_\alpha([P]) \)
- \( [[\alpha]P] = \delta_\alpha([P]) \)

---

Only for HPs. No interactive play!

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Summary

Semantics for differential game logic
- Simple compositional denotational semantics
  - Meaning is a simple function of its pieces
  - Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint

Next chapter
1. Axiomatics
2. How to win and prove hybrid games
André Platzer.
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André Platzer.
Differential game logic.