15: Winning Strategies & Regions
Logical Foundations of Cyber-Physical Systems

André Platzer
Outline

1 Learning Objectives

2 Denotational Semantics
   - Differential Game Logic Semantics
   - Hybrid Game Semantics

3 Semantics of Repetition
   - Repetition with Advance Notice
   - Infinite Iterations and Inflationary Semantics
   - Ordinals
   - Inflationary Semantics of Repetitions
   - Implicit Definitions vs. Explicit Constructions
   - +1 Argument
   - Fixpoints and Pre-fixpoints
   - Comparing Fixpoints
   - Characterizing Winning Repetitions Implicitly

4 Summary
Outline

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4. Summary
Learning Objectives
Winning Strategies & Regions

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
denotational vs. operational semantics

adversarial dynamics
adversarial semantics
adversarial repetitions
fixpoints

CPS semantics
multi-agent operational-effects
mutual reactions
complementary hybrid systems

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### Differential Game Logic: Syntax

**Definition (Hybrid game $\alpha$)**

\[
x := e \mid ?Q \mid x' = f(x) \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d
\]

**Definition (dGL Formula $P$)**

\[
p(e_1, \ldots, e_n) \mid e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid \langle \alpha \rangle P \mid [\alpha]P
\]
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4. Summary
### Definition (dGL Formula $P$)

<table>
<thead>
<tr>
<th>Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[ e_1 \geq e_2 ]$</td>
<td>${ \omega \in S : \omega[e_1] \geq \omega[e_2] }$</td>
</tr>
<tr>
<td>$[\neg P]$</td>
<td>$([P])^c$</td>
</tr>
<tr>
<td>$[ P \land Q ]$</td>
<td>$[P] \cap [Q]$</td>
</tr>
<tr>
<td>$[\langle \alpha \rangle P]$</td>
<td>$\varsigma_\alpha([P]) { \omega : \nu \in [P] \text{ for some } \nu \text{ with } (\omega, \nu) \in [\alpha] }$</td>
</tr>
<tr>
<td>$[[\alpha]P]$</td>
<td>$\delta_\alpha([P])$</td>
</tr>
</tbody>
</table>
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_x := e(X) = \{ \omega \in S : \omega \omega [e] x \in X \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_x := e(X) = \{ \omega \in S : \omega_x^e \in X \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{x'} = f(x)(X) = \{ \phi(0) \in S : \phi(r) \in X, d\phi(t)(x) = \phi(z)[f(x)] \text{ for all } z \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{x'=f(x)}(X) = \{ \varphi(0) \in S : \varphi(r) \in X, \frac{d\varphi(t)(x)}{dt}(z) = \varphi(z)[f(x)] \text{ for all } z \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\mathcal{S}_Q(X) =$$
Differential Game Logic: Denotational Semantics

Definition (Hybrid game $\alpha$: denotational semantics)

$\mathcal{?v}Q(X) = \llbracket Q \rrbracket \cap X$

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Definition (Hybrid game $\alpha$: denotational semantics)

\[ s_{\alpha \cup \beta}(X) = \]

\[ s_\alpha(X) \cup s_\beta(X) \]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\mathcal{S}_{\alpha \cup \beta}(X) = \mathcal{S}_\alpha(X) \cup \mathcal{S}_\beta(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{\alpha;\beta}(X) = \text{[Diagram]}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{\alpha;\beta}(X) = s_\alpha(s_\beta(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha^d}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varrho_{\alpha^d}(X) =$$

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Definition (Hybrid game $\alpha$: denotational semantics)

$$s_{\alpha^d}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^C))^C$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x=e(X)} =$$
Definition (Hybrid game $\alpha$: denotational semantics)

\[ \delta_{x:=e}(X) = \{ \omega \in S : \omega_x^{[e]} \in X \} \]
Definition (Hybrid game $\alpha$: denotational semantics)

\[ \delta_{x'} = f(x)(X) = \]

\[ x' = f(x)(X) \]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{x' = f(x)}(X) = \{ \varphi(0) \in S : \varphi(z) \in X, \frac{d\varphi(t)(x)}{dt}(z) = \varphi(z)[f(x)] \text{ for all } z \}$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta \sqsubseteq_Q (X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta Q(X) = [Q]^c \cup X$$
Definition (Hybrid game $\alpha$: denotational semantics)

\[ \delta_{\alpha \cup \beta}(X) = \]

\[ \delta_{\alpha}(X) \cap \delta_{\beta}(X) \]
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha \cup \beta}(X) = \delta_{\alpha}(X) \cap \delta_{\beta}(X)$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha;\beta}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha;\beta}(X) = \delta_\alpha(\delta_\beta(X))$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha^\sigma}(X) =$$
Definition (Hybrid game $\alpha$: denotational semantics)

$$\delta_{\alpha^d}(X) = (\delta_{\alpha}(X^C))^C$$
### Differential Game Logic: Denotational Semantics

**Definition (Hybrid game \( \alpha \))**

\[
\begin{align*}
\varsigma_x := e(X) &= \{ \omega \in S : \omega_x^{[e]} \in X \} \\
\varsigma_x' := f(x)(X) &= \{ \varphi(0) \in S : \varphi(r) \in X, \quad \frac{d\varphi(t)(x)}{dt}(z) = \varphi(z)^{[f(x)]} \text{ for all } z \} \\
\varsigma ? Q(X) &= [Q] \cap X \\
\varsigma \cup \beta(X) &= \varsigma \alpha(X) \cup \varsigma \beta(X) \\
\varsigma \alpha ; \beta(X) &= \varsigma \alpha(\varsigma \beta(X)) \\
\varsigma \alpha^*(X) &= \\
\varsigma \alpha^d(X) &= (\varsigma \alpha(X^c))^c
\end{align*}
\]

**Definition (dGL Formula \( P \))**

\[
\begin{align*}
[e_1 \geq e_2] &= \{ \omega \in S : \omega^{[e_1]} \geq \omega^{[e_2]} \} \\
[\neg P] &= ([P])^c \\
[P \land Q] &= [P] \cap [Q] \\
[\langle \alpha \rangle P] &= \varsigma \alpha([P]) \\
[[\alpha] P] &= \delta \alpha([P])
\end{align*}
\]
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4 Summary
\[(x := 0 \land x := 1)^* x = 0\]

\[
\text{wfd} \quad \iff \text{false unless } x = 0
\]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$\mathcal{S}_{\alpha^*}(X) =$
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$

$$\llbracket \alpha^* \rrbracket = \bigcup_{n \in \mathbb{N}} \llbracket \alpha^n \rrbracket \quad \text{where} \quad \alpha^{n+1} \equiv \alpha^n; \quad \alpha^0 \equiv \text{true} \quad \text{for HP } \alpha$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha^n}(X)$$
Definition (Hybrid game $\alpha$)

$\mathcal{s}_\alpha^*(X) = \bigcup_{n \in \mathbb{N}} \mathcal{s}_\alpha^n(X)$

Advance notice semantics
Note (+1 argument)

\[ Y \subseteq \mathcal{s}_\alpha^*(X) \text{ then } \mathcal{s}_\alpha(Y) \subseteq \mathcal{s}_\alpha^*(X) \]

Since \( \mathcal{s}_\alpha(Y) \) is just one round away from \( Y \).

\[ \mathcal{s}_\alpha(Y) \setminus \mathcal{s}_\alpha^*(X) \]

\[ \emptyset \]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma^{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} \varsigma^{n\alpha}(X)$$

$$\begin{align*}
\varsigma^0_\alpha(X) & \overset{\text{def}}{=} X \\
\varsigma^{\kappa+1}_\alpha(X) & \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma^\kappa_\alpha(X))
\end{align*}$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s_{\alpha}^n(X)$$

$$s_{\alpha}^0(X) \overset{\text{def}}{=} X$$

$$s_{\alpha}^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s_{\alpha}^\kappa(X))$$

Example

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$$
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{n \in \mathbb{N}} s^n_{\alpha}(X)$$

$$s^0_{\alpha}(X) \overset{\text{def}}{=} X$$

$$s^{\kappa+1}_{\alpha}(X) \overset{\text{def}}{=} X \cup s_{\alpha}(s^\kappa_{\alpha}(X))$$

Example

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$$

$$s^n_{\alpha}([0, 1)) = [0, n) \neq \mathbb{R}$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s^*_\alpha(X) = \bigcup_{n \in \mathbb{N}} s^n_\alpha(X)$$

$$s^0_\alpha(X) \overset{\text{def}}{=} X$$

$$s^{\kappa+1}_\alpha(X) \overset{\text{def}}{=} X \cup s_\alpha(s^\kappa_\alpha(X))$$

$$s^\lambda_\alpha(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s^\kappa_\alpha(X)$$

$\lambda \neq 0$ a limit ordinal

Example

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1)$$

$$s^n_\alpha([0, 1)) = [0, n) \neq \mathbb{R}$$

$$s^\omega_\alpha([0, 1)) = \bigcup_{n \in \mathbb{N}} s^n_\alpha([0, 1)) = [0, \infty) \neq \mathbb{R}$$
Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$s_\alpha^*(X) = \bigcup_{\kappa<\infty} s_\alpha^\kappa(X)$$

$$s_\alpha^0(X) \overset{\text{def}}{=} X$$

$$s_\alpha^{\kappa+1}(X) \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^\kappa(X))$$

$$s_\alpha^\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa<\lambda} s_\alpha^\kappa(X) \quad \lambda \neq 0 \text{ a limit ordinal}$$

**Example**

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle (0 \leq x < 1) \quad s_\alpha^n([0, 1)) = [0, n) \neq \mathbb{R}$$

$$s_\alpha^{\omega+1}([0, 1)) = s_\alpha([0, \infty)) = \mathbb{R} \quad s_\alpha^{\omega}([0, 1)) = \bigcup_{n \in \mathbb{N}} s_\alpha^n([0, 1)) = [0, \infty) \neq \mathbb{R}$$
Theorem

Hybrid game closure ordinal $> \omega^\omega$
Expedition: Ordinal Arithmetic

\[ \iota + 0 = \iota \]
\[ \iota + (\kappa + 1) = (\iota + \kappa) + 1 \text{ successor } \kappa + 1 \]
\[ \iota + \lambda = \bigsqcup_{\kappa < \lambda} \iota + \kappa \text{ limit } \lambda \]
\[ \iota \cdot 0 = 0 \]
\[ \iota \cdot (\kappa + 1) = (\iota \cdot \kappa) + \iota \text{ successor } \kappa + 1 \]
\[ \iota \cdot \lambda = \bigsqcup_{\kappa < \lambda} \iota \cdot \kappa \text{ limit } \lambda \]
\[ \iota^0 = 1 \]
\[ \iota^{\kappa + 1} = \iota^\kappa \cdot \iota \text{ successor } \kappa + 1 \]
\[ \iota^\lambda = \bigsqcup_{\kappa < \lambda} \iota^\kappa \text{ limit } \lambda \]

\[ 2 \cdot \omega = 4 \cdot \omega \neq \omega \cdot 2 < \omega \cdot 4 \]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s^\alpha_\kappa(X) = \bigcup_{\kappa < \omega} s^{\kappa}_\alpha(X)$$

- $s^\alpha_0(X) \overset{\text{def}}{=} X$
- $s^\alpha_{\kappa+1}(X) \overset{\text{def}}{=} X \cup s^\alpha_\kappa(s^\kappa_\alpha(X))$
- $s^\alpha_\lambda(X) \overset{\text{def}}{=} \bigcup_{\kappa < \lambda} s^\kappa_\alpha(X) \quad \lambda \neq 0 \text{ a limit ordinal}$
Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_\kappa(X)$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X)$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$s_{\alpha^*}(X) = \bigcup_{\kappa < \infty} s_{\alpha}^{\kappa}(X)$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcup_{\kappa < \infty} \varsigma_\alpha^\kappa(X)$$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$$\varsigma_{\alpha^*}(X) = \bigcup_{\kappa < \infty} \varsigma_{\kappa}(X)$$
The advantages of implicit definition over construction are roughly those of theft over honest toil.

— Bertrand Russell
Y \subseteq \alpha^*(X) \text{ then } \alpha(Y) \subseteq \alpha^*(X)

Since \alpha(Y) is just one round away from Y.
Note (+1 argument)

\[ Y \subseteq \kappa_\alpha^*(X) \text{ then } \kappa_\alpha(Y) \subseteq \kappa_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} \kappa_\alpha^*(X) \text{ then } \kappa_\alpha(Z) \subseteq \kappa_\alpha^*(X) = Z \]
Note (+1 argument)

$$Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X)$$

$$Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z$$

- Which $Z$ with $s_\alpha(Z) \subseteq Z$ is the right one?
- Are there multiple such $Z$?
- Does such a $Z$ exist?
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

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- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
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- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( s_\alpha Q^d(\emptyset) = s_\alpha Q(\emptyset \setminus S) = (\lceil Q \rceil \cap S) \supseteq [Q] \subseteq \emptyset \)
+1 Argument

Note (+1 argument)

\[ Y \subseteq s_\alpha^*(X) \text{ then } s_\alpha(Y) \subseteq s_\alpha^*(X) \]

\[ Z \overset{\text{def}}{=} s_\alpha^*(X) \text{ then } s_\alpha(Z) \subseteq s_\alpha^*(X) = Z \]

- Which \( Z \) with \( s_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( s_\beta Q_d(\emptyset) = s_\beta Q(\emptyset^c)^c = (\llbracket Q \rrbracket \cap S)^c = \llbracket Q \rrbracket^c \nsubseteq \emptyset \)
- Then: \( s_\beta Q_d([\neg Q]) = s_\beta Q([\neg Q]^c)^c = ([\llbracket Q \rrbracket \cap [Q])^c = [\neg Q] \subseteq [\neg Q] \)
Note (+1 argument)

\( Y \subseteq \mathcal{s}_\alpha^*(X) \) then \( \mathcal{s}_\alpha(Y) \subseteq \mathcal{s}_\alpha^*(X) \)

\( Z \overset{\text{def}}{=} \mathcal{s}_\alpha^*(X) \) then \( \mathcal{s}_\alpha(Z) \subseteq \mathcal{s}_\alpha^*(X) = Z \)

- Which \( Z \) with \( \mathcal{s}_\alpha(Z) \subseteq Z \) is the right one?
- Are there multiple such \( Z \)?
- Does such a \( Z \) exist?
- Existence: \( Z = \emptyset \)
- No wait, dual tests: \( \mathcal{s}_?Q^d(\emptyset) = \mathcal{s}_?Q(\emptyset^C)^C = (\llbracket Q \rrbracket \cap \mathcal{S})^C = \llbracket Q \rrbracket^C \not\subseteq \emptyset \)
- Then: \( \mathcal{s}_?Q^d([\neg Q]) = \mathcal{s}_?Q([\neg Q]^C)^C = ([\llbracket Q \rrbracket \cap [Q])^C = [\neg Q] \subseteq [\neg Q] \)
- Still too small: \( X \subseteq Z \) since Angel may decide not to repeat
Fixpoints and Pre-Fixpoints

**Definition (Pre-fixpoint)**

\[ X \cup \varsigma_\alpha(Z) \subseteq Z \]

for the winning region \( Z \overset{\text{def}}{=} \varsigma_\alpha^*(X) \)

Which \( Z \) is the right one? Are there multiple such \( Z \)? Does such a \( Z \) exist?

Existence: \( Z = S \) but that's too big and independent of \( \alpha \)

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Fixpoints and Pre-Fixpoints

Definition (Pre-fixpoint)

\[ X \cup \mathcal{s}_\alpha(Z) \subseteq Z \quad \text{for the winning region } Z \overset{\text{def}}{=} \mathcal{s}_\alpha^*(X) \]

- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
Definition (Pre-fixpoint)

\[ X \cup \mathcal{S}_\alpha(Z) \subseteq Z \]
for the winning region \( Z \stackrel{\text{def}}{=} \mathcal{S}_{\alpha^*}(X) \)

- Which \( Z \) is the right one?
- Are there multiple such \( Z \)? Does such a \( Z \) exist?
- Existence: \( Z = S \)
Fixpoints and Pre-Fixpoints

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- Are there multiple such \( Z \)? Does such a \( Z \) exist?
- Existence: \( Z = S \) but that's too big and independent of \( \alpha \)
Comparing (Pre-)Fixpoints

**Lemma ( )**

\[ X \cup s_\alpha(Y) \subseteq Y \]
\[ X \cup s_\alpha(Z) \subseteq Z \]

are pre-fixpoints, then

\[ Y \cap Z \] is a smaller pre-fixpoint.

Proof.

\[ X \cup s_\alpha(Y \cap Z) \] mon \[ \subseteq X \cup (s_\alpha(Y) \cap s_\alpha(Z)) \] above \[ \subseteq Y \cap Z \]

Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!

So: repetition semantics is the smallest pre-fixpoint (well-founded)
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

\[ X \cup \mathcal{S}_\alpha(Y) \subseteq Y \]
\[ X \cup \mathcal{S}_\alpha(Z) \subseteq Z \]

are pre-fixpoints, then \( Y \cap Z \) is a smaller pre-fixpoint.
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

\[ X \cup \varsigma(\alpha)(Y) \subseteq Y \]
\[ X \cup \varsigma(\alpha)(Z) \subseteq Z \]

are pre-fixpoints, then \( Y \cap Z \) is a smaller pre-fixpoint.

Proof.

\[ X \cup \varsigma(\alpha)(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\varsigma(\alpha)(Y) \cap \varsigma(\alpha)(Z)) \overset{\text{above}}{\subseteq} Y \cap Z \]
Comparing (Pre-)Fixpoints

**Lemma (Intersection closure)**

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\[ X \cup s_\alpha(Y \cap Z)^{\text{mon}} \subseteq X \cup (s_\alpha(Y) \cap s_\alpha(Z))^{\text{above}} \subseteq Y \cap Z \]

Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!
Comparing (Pre-)Fixpoints

Lemma (Intersection closure)

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\[ X \cup \mathcal{S}_\alpha(Y \cap Z) \overset{\text{mon}}{\subseteq} X \cup (\mathcal{S}_\alpha(Y) \cap \mathcal{S}_\alpha(Z)) \overset{\text{above}}{\subseteq} Y \cap Z \]

Even: The intersection of any family of pre-fixpoints is a pre-fixpoint!
So: repetition semantics is the smallest pre-fixpoint (well-founded)
Definition (Hybrid game $\alpha$)

$$s_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z \}$$
Semantics of Repetition

**Definition (Hybrid game $\alpha$)**

$$s_\alpha^*(X) = \bigcap \{Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z\}$$

$X \cup s_\alpha(s_\alpha^*(X)) \subseteq s_\alpha^*(X)$

$s_\alpha^*(X)$ intersection of solutions
Definition (Hybrid game $\alpha$)

$$\varsigma_\alpha^*(X) = \bigcap \{Z \subseteq S : X \cup \varsigma_\alpha(Z) \subseteq Z\}$$

$Z \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha(Z) \subseteq \varsigma_\alpha(\varsigma_\alpha^*(X))$  \hspace{1cm} $\varsigma_\alpha^*(X)$ intersection of solutions by mon since $Z \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha^*(X)$ intersection of solutions by mon since $Z \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha^*(X) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$
Definition (Hybrid game $\alpha$)

$\varsigma_{\alpha^*}(X) = \bigcap \{Z \subseteq S : X \cup \varsigma_{\alpha}(Z) \subseteq Z\}$

$Z \overset{\text{def}}{=} X \cup \varsigma_{\alpha}(\varsigma_{\alpha^*}(X)) \subseteq \varsigma_{\alpha^*}(X)$

$X \cup \varsigma_{\alpha}(Z) \subseteq X \cup \varsigma_{\alpha}(\varsigma_{\alpha^*}(X)) = Z$ by mon since $Z \subseteq \varsigma_{\alpha^*}(X)$
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$s_{\alpha^*}(X) = \bigcap\{Z \subseteq S : X \cup s_\alpha(Z) \subseteq Z\}$

$Z \overset{\text{def}}{=} X \cup s_\alpha(s_{\alpha^*}(X)) \subseteq s_{\alpha^*}(X)$

$X \cup s_\alpha(Z) \subseteq X \cup s_\alpha(s_{\alpha^*}(X)) = Z$ by mon since $Z \subseteq s_{\alpha^*}(X)$

$s_{\alpha^*}(X) \subseteq X \cup s_\alpha(s_{\alpha^*}(X)) = Z$ since $s_{\alpha^*}(X)$ smallest such $Z$
Semantics of Repetition

**Definition (Hybrid game \( \alpha \))**

\[
\sigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \sigma_\alpha(Z) \subseteq Z \}
\]

\( \sigma_\alpha^* \) intersection of solutions

\[
Z \overset{\text{def}}{=} X \cup \sigma_\alpha(\sigma_\alpha^*(X)) \subseteq \sigma_\alpha^*(X)
\]

\( X \cup \sigma_\alpha(Z) \subseteq X \cup \sigma_\alpha(\sigma_\alpha^*(X)) = Z \) by mon since \( Z \subseteq \sigma_\alpha^*(X) \)

\( \sigma_\alpha^*(X) \subseteq X \cup \sigma_\alpha(\sigma_\alpha^*(X)) = Z \) since \( \sigma_\alpha^*(X) \) smallest such \( Z \)
**Semantics of Repetition**

**Definition (Hybrid game $\alpha$)**

\[ s_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup s_\alpha(Z) = Z \} \]

\[ s_\alpha^*(X) \subseteq s_\alpha^*(X) \]

\[ s_\alpha^*(X) \subseteq X \cup s_\alpha(s_\alpha^*(X)) \]

\[ s_\alpha^*(X) = X \cup s_\alpha(s_\alpha^*(X)) = Z \]

\[ s_\alpha^*(X) \text{ intersection of solutions} \]

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\[ s_\alpha^*(X) \text{ intersection of solutions} \]

**Z** \( \overset{\text{def}}{=} \)

\[ Z \overset{\text{def}}{=} X \cup s_\alpha(s_\alpha^*(X)) \subseteq s_\alpha^*(X) \]

\[ X \cup s_\alpha(Z) \subseteq X \cup s_\alpha(s_\alpha^*(X)) = Z \text{ by mon since } Z \subseteq s_\alpha^*(X) \]

\[ s_\alpha^*(X) = X \cup s_\alpha(s_\alpha^*(X)) = Z \text{ since } s_\alpha^*(X) \text{ smallest such } Z \]
Semantics of Repetition

Definition (Hybrid game $\alpha$)

$\varsigma_\alpha^*(X) = \bigcap \{ Z \subseteq S : X \cup \varsigma_\alpha(Z) = Z \} = \bigcup_{\kappa < \infty} \varsigma_\alpha^\kappa(X)$ by Knaster-Tarski

$\varsigma_\alpha^*(X) = X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$

$X \cup \varsigma_\alpha(Z) \subseteq X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ by mon since $Z \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha^*(X) = X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) = Z$ since $\varsigma_\alpha^*(X)$ smallest such $Z$

$Z \overset{\text{def}}{=} X \cup \varsigma_\alpha(\varsigma_\alpha^*(X)) \subseteq \varsigma_\alpha^*(X)$

$\varsigma_\alpha^*(X)$ intersection of solutions
Outline

1. Learning Objectives

2. Denotational Semantics
   • Differential Game Logic Semantics
   • Hybrid Game Semantics

3. Semantics of Repetition
   • Repetition with Advance Notice
   • Infinite Iterations and Inflationary Semantics
   • Ordinals
   • Inflationary Semantics of Repetitions
   • Implicit Definitions vs. Explicit Constructions
   • $+1$ Argument
   • Fixpoints and Pre-fixpoints
   • Comparing Fixpoints
   • Characterizing Winning Repetitions Implicitly

4. Summary
Differential Game Logic: Denotational Semantics

**Definition (Hybrid game $\alpha$)**

\[
[s_x := e(x)] = \{ \omega \in S : \omega_e^\omega \in X \}
\]

\[
s_{x'} = f(x)(x) = \{ \varphi(0) \in S : \varphi(r) \in X, \frac{d\varphi(t)(x)}{dt}(z) = \varphi(z)[f(x)] \text{ for all } z \}
\]

\[
s?Q(x) = [Q] \cap X
\]

\[
s_\alpha \cup_\beta (x) = s_\alpha(x) \cup s_\beta(x)
\]

\[
s_\alpha ;_\beta (x) = s_\alpha(s_\beta(x))
\]

\[
s_\alpha^* (x) = \bigcup_{\kappa < \infty} s_\alpha^\kappa(x)
\]

\[
s_\alpha^d (x) = (s_\alpha(X^C))^C
\]

**Definition (dGL Formula $P$)**

\[
[e_1 \geq e_2] = \{ \omega \in S : \omega[e_1] \geq \omega[e_2] \}
\]

\[
[\neg P] = (\|P\|)^C
\]

\[
[P \land Q] = [P] \cap [Q]
\]

\[
[\langle \alpha \rangle P] = s_\alpha([P])
\]

\[
[[\alpha] P] = \delta_\alpha([P])
\]
### Differential Game Logic: Denotational Semantics

**Definition (Hybrid game $\alpha$)**

$\mathcal{S}_x := e(X) = \{\omega \in S : \omega_x[e] \in X\}$

$\mathcal{S}_x' = f(x)(X) = \{\phi(0) \in S : \phi(r) \in X, \frac{d \phi(t)(x)}{dt}(z) = \phi(z)[f(x)]\text{ for all } z\}$

$\mathcal{S}?Q(X) = \mathcal{[Q]} \cap X$

$\mathcal{S}_\alpha \cup \beta(X) = \mathcal{S}_\alpha(X) \cup \mathcal{S}_\beta(X)$

$\mathcal{S}_\alpha ; \beta(X) = \mathcal{S}_\alpha(\mathcal{S}_\beta(X))$

$\mathcal{S}_\alpha^*(X) = \bigcap\{Z \subseteq S : X \cup \mathcal{S}_\alpha(Z) \subseteq Z\}$

$\mathcal{S}_\alpha^d(X) = (\mathcal{S}_\alpha(X^c))^c$

**Definition (dGL Formula $P$)**

$\mathcal{[e_1 \geq e_2]} = \{\omega \in S : \omega[e_1] \geq \omega[e_2]\}$

$\mathcal{[\neg P]} = (\mathcal{[P]})^c$

$\mathcal{[P \land Q]} = \mathcal{[P]} \cap \mathcal{[Q]}$

$\mathcal{[\langle \alpha \rangle P]} = \mathcal{S}_\alpha(\mathcal{[P]})$

$\mathcal{[[\alpha] P]} = \mathcal{\delta}_\alpha(\mathcal{[P]})$
Summary

- Semantics for differential game logic
- Simple compositional denotational semantics
- Meaning is a simple function of its pieces
- Outlier: repetition is subtle higher-ordinal iteration
- Better: repetition means least fixpoint

Next chapter
1. Axiomatics
2. How to win and prove hybrid games