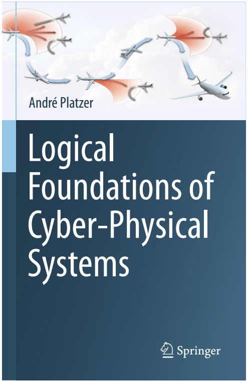
09: Reactions & Delays

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

Outline

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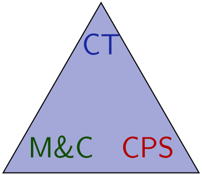
Learning Objectives

using loop invariants
design time-triggered control
design-by-invariant



modeling CPS
designing controls
time-triggered control
reaction delays
discrete sensing

semantics of time-triggered control
operational effect
finding control constraints
model-predictive control

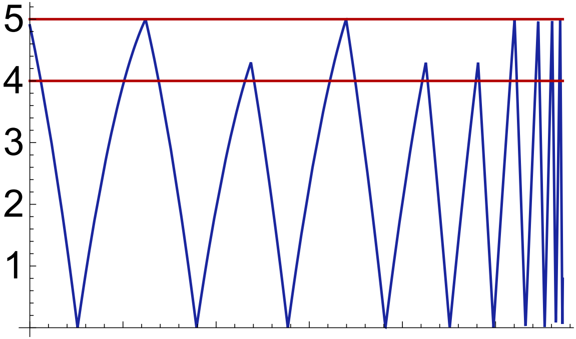
Outline

1. Learning Objectives
2. Delays in Control
   - The Impact of Delays on Event Detection
   - Cartesian Demon
   - Model-Predictive Control Basics
   - Design-by-Invariant
   - Controlling the Control Points
   - Sequencing and Prioritizing Reactions
   - Time-Triggered Verification
3. Summary

Proposition (Quantum can play ping-pong safely)

$$0 \le x \land x \le 5 \land v \le 0 \land g > 0 \land 1 \ge c \ge 0 \land f \ge 0 \rightarrow$$
$$\big[\big((\{x' = v, v' = -g \,\&\, x \ge 0 \land x \le 5\} \cup \{x' = v, v' = -g \,\&\, x \ge 5\});$$
$$\text{if}(x=0)\ v := -cv \text{ else if}(4 \le x \le 5 \land v \ge 0)\ v := -fv\big)^*\big](0 \le x \le 5)$$

Proof @invariant$(0 \le x \le 5 \land (x = 5 \rightarrow v \le 0))$



Quantum's Ping-Pong Proof Invariants

Proposition (Quantum can play ping-pong safely)

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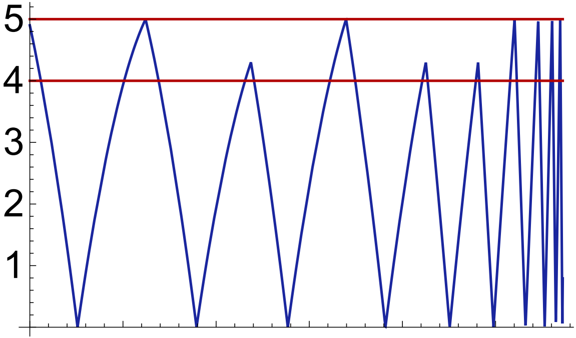

Just can't implement ...

Physical vs. Controller Events

1. Justifiable: Physical events (on ground $x = 0$)
2. Justifiable: Physical evolution domains (above ground $x \geq 0$)
3. Questionable: Controller evolution domain ($x \leq 5$)
4. Unlike physics, controllers won't run *all* the time. Just fairly often.
5. Controllers cannot sense and compute all the time.

If you expect the world to change for your controller's sake, you may be in for a surprise.

Conjecture (Quantum can play ping-pong safely)

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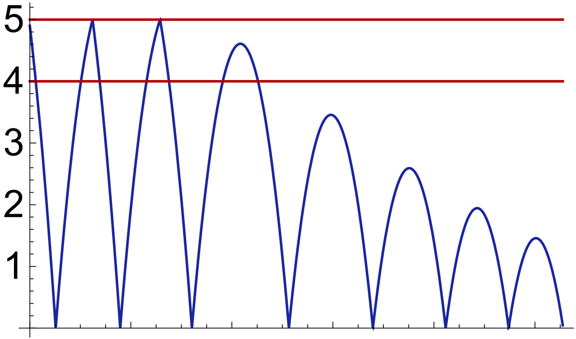
Proof? Ask René Descartes



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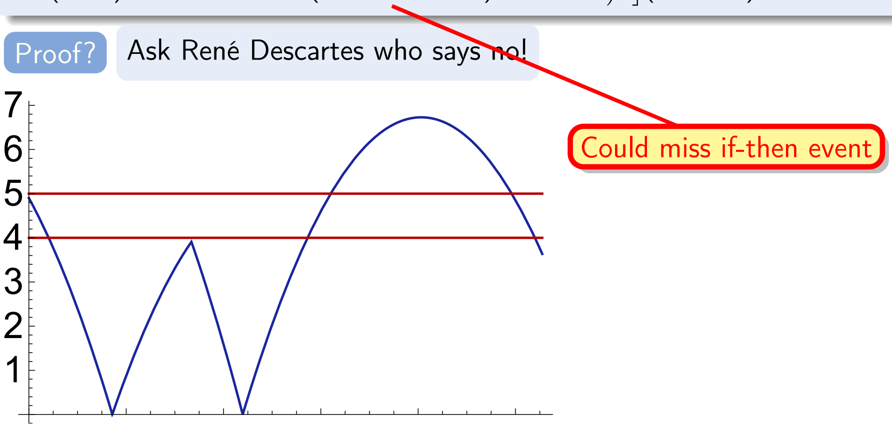

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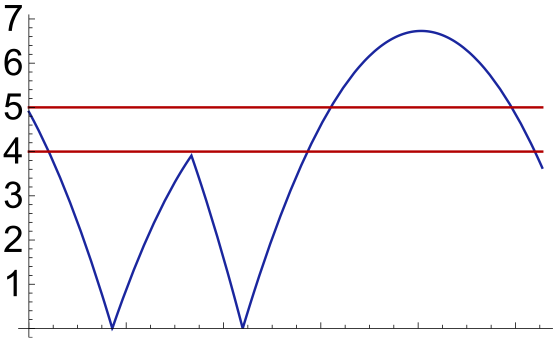

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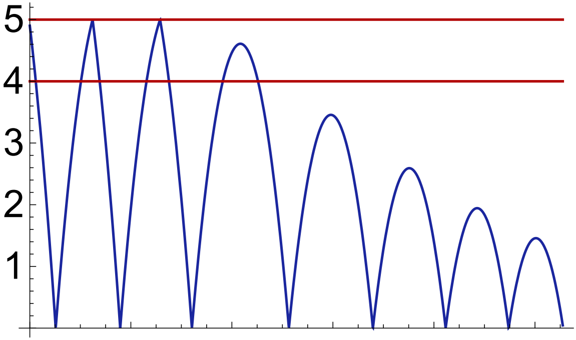

Back to the Drawing Desk: Quantum the Ping-Pong Ball

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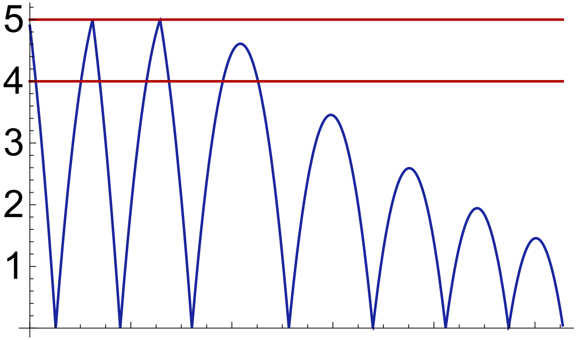

Wind up a clock

Quantum the Time-triggered Ping-Pong Ball

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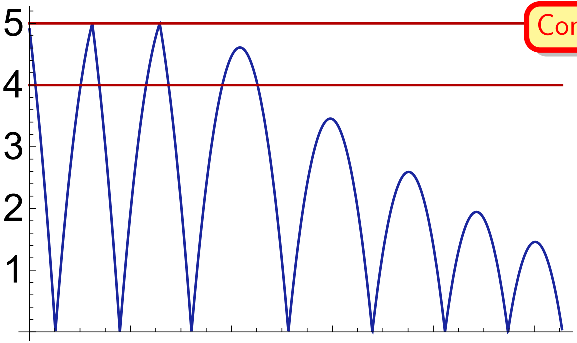
Control action before physics



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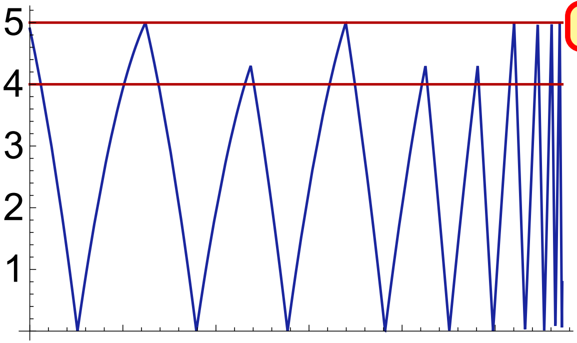

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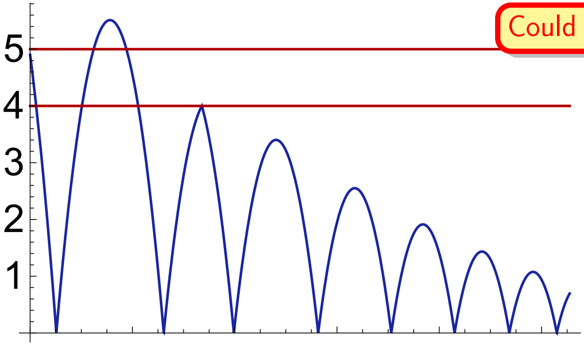
Could miss event off control cycle



Delays vs. Events

1. Periodically/frequently monitor for an event with a polling frequency / reaction time.
2. Delays may make the controller miss events.
3. Discrepancy between event-triggered idea vs. real time-triggered implementation.
4. Issues indicate poor event abstraction.
5. Slow controllers monitoring small regions of a fast moving system.
6. Controller needs to be aware of its own delay.

Cartesian Doubt: René Descartes's Cartesian Demon 1641

Outwit the Cartesian Demon

Skeptical about the truth of all beliefs until justification has been found.

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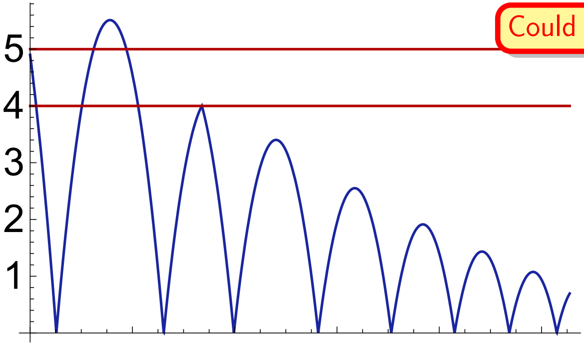

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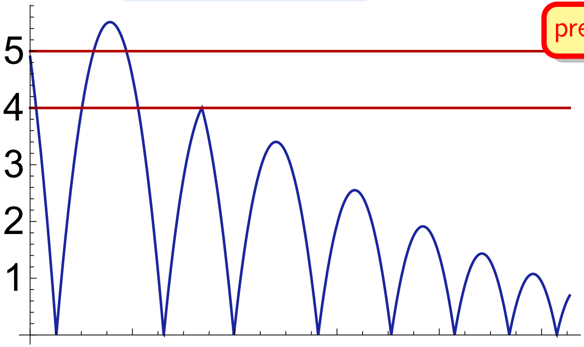
Proof? Ask René Descartes

predict 1s: $x + v - \frac{g}{2} > 5$



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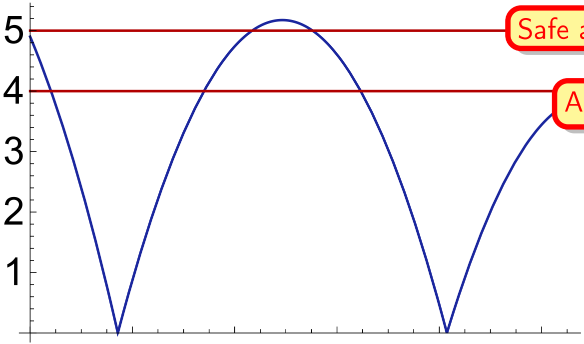

Safe after 1 s but not until then

All depends on sampling

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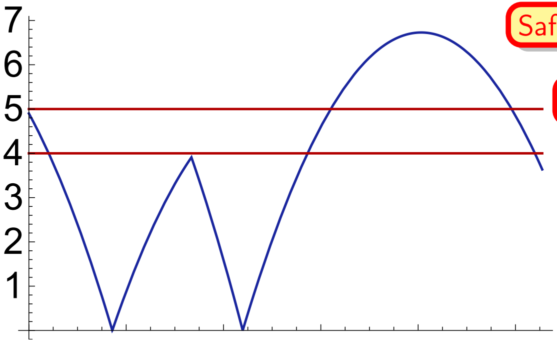

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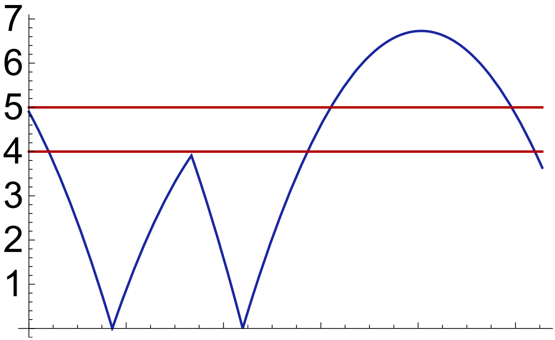
Design-by-Invariant

$2gx = 2gH - v^2 \land x \geq 0 \ \land c = 1 \land g > 0$

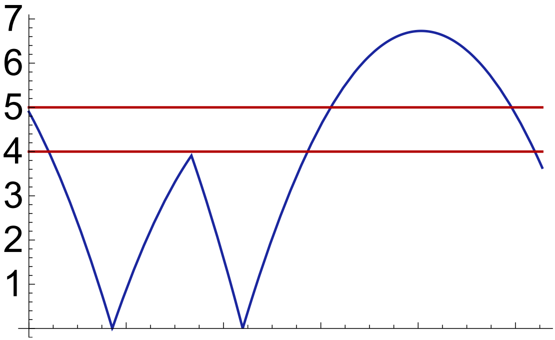
bouncing ball invariant



Design-by-Invariant

$2gx = 2gH - v^2 \land x \geq 0 \ \land c = 1 \land g = 1$

simplify arithmetic



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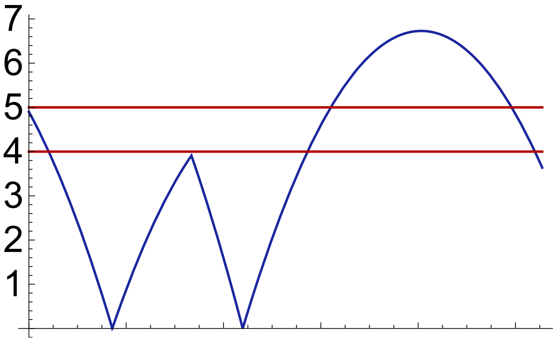
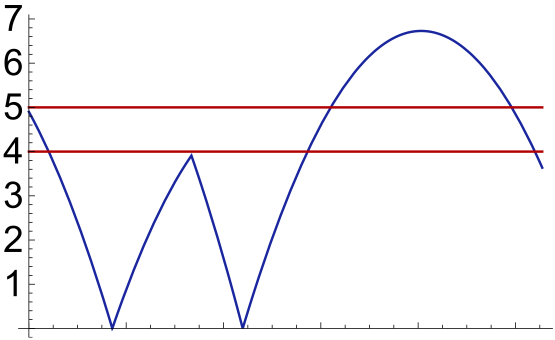


Design-by-Invariant

$$2x = 2 \cdot H - v^2 \wedge x \geq 0$$

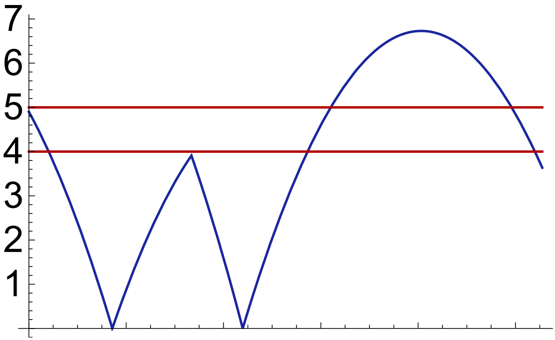


Quantum Discovers Design-by-Invariant

Design-by-Invariant

$2x = 2 \cdot 5 - v^2 \wedge x \geq 0$

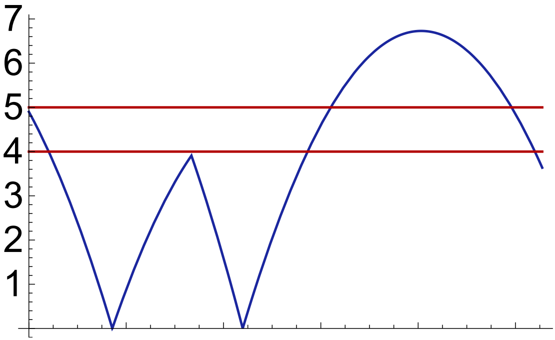
critical height



Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \land x \geq 0$

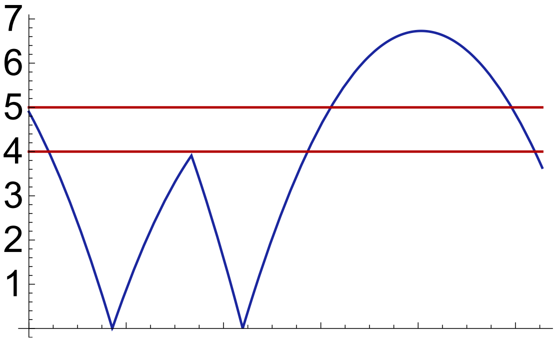
potential exceeds safe height



Design-by-Invariant

$2x > 2 \cdot 5 - v^2 \land x \geq 0$

use invariant for control



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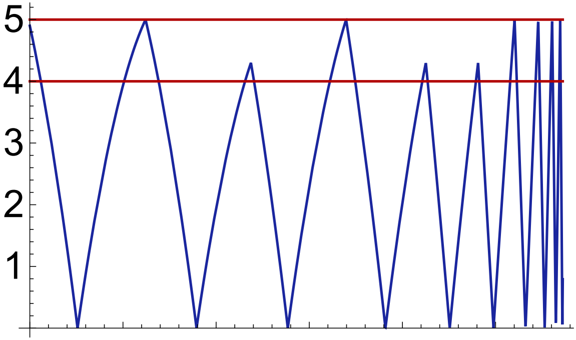
Proof? Ask René Descartes



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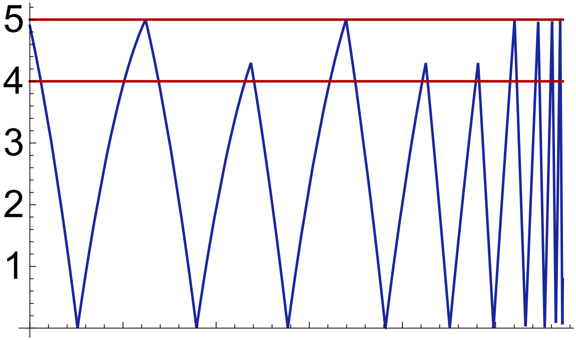

Just for simplicity

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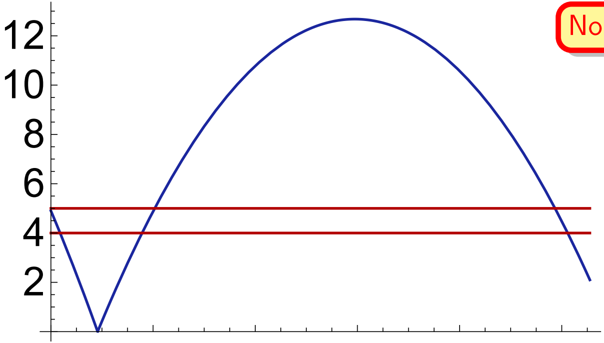
No control near ground



Quantum the Time-triggered Ping-Pong Ball

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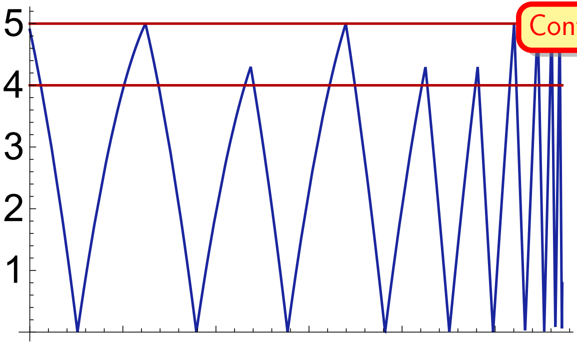
Control despite ground



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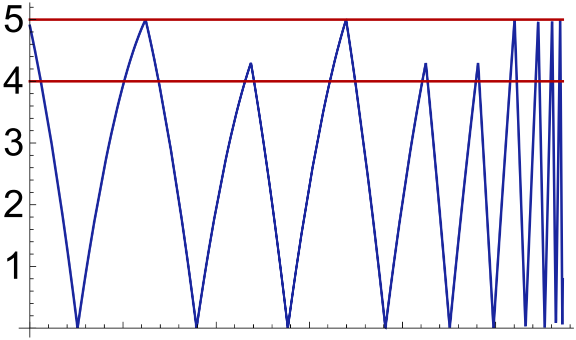

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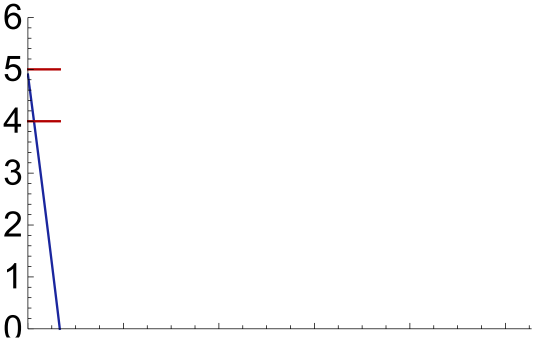

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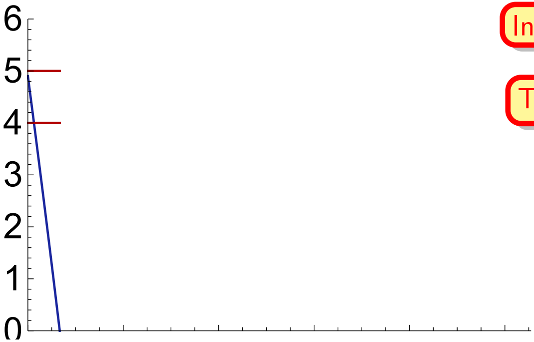

Invariants are **invariants!**

True ever $\rightsquigarrow$ true always

Quantum the Time-triggered Ping-Pong Ball

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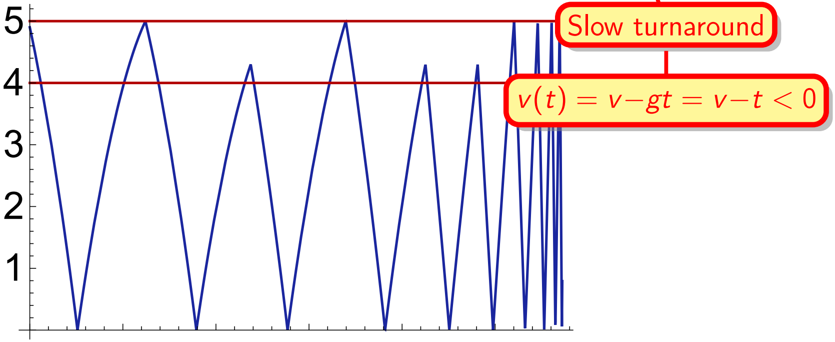
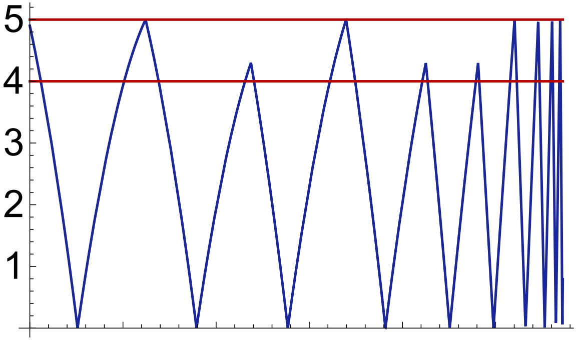


Conjecture (Quantum can play ping-pong safely)

$$0 \le x \wedge x \le 5 \wedge v \le 0 \wedge g = 1 \wedge 1 = c \wedge 1 = f \rightarrow$$
$$\Big[\big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\tfrac{1}{2}{-}v \vee 2x{>}2{\cdot}5{-}v^2{\wedge}v{<}1){\wedge}v{\ge}0)\ v := -fv;$$
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \wedge t{\le}1\}\big)^*\Big](0{\le}x{\le}5)$$

Proof? Ask René Descartes who says yes



Proposition (Quantum can play ping-pong safely in real-time)

$$0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \rightarrow$$
$$\Big[\Big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\tfrac{1}{2}{-}v \lor 2x{>}2{\cdot}5{-}v^2{\land}v{<}1){\land}v{\ge}0)\ v := -fv;$$
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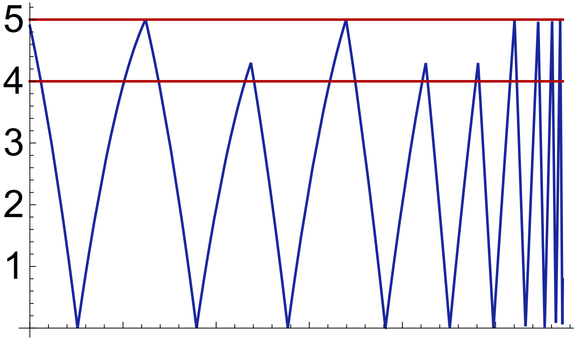
Proof



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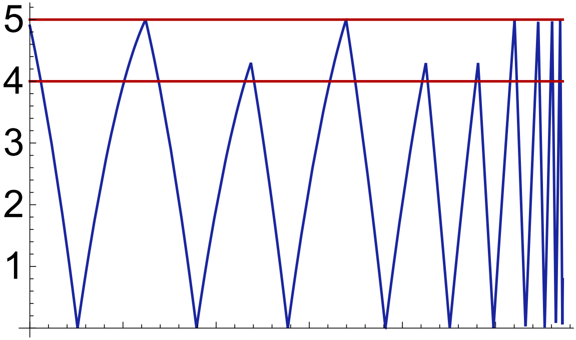
Proof **@invariant**$(2x = 2H - v^2 \land x \ge 0 \land x{\le}5)$



Proposition (Quantum can play ping-pong safely in real-time)

$2x = 2H - v^2 \wedge 0 \leq x \wedge x \leq 5 \wedge v \leq 0 \wedge g{=}1{>}0 \wedge 1{=}c{\geq}0 \wedge 1{=}f{\geq}0 \rightarrow$
$\big[\big(\text{if}(x{=}0)\ v := -cv; \text{if}((x{>}5\tfrac{1}{2}{-}v \vee 2x{>}2{\cdot}5{-}v^2{\wedge}v{<}1){\wedge}v{\geq}0)\ v := -fv;$
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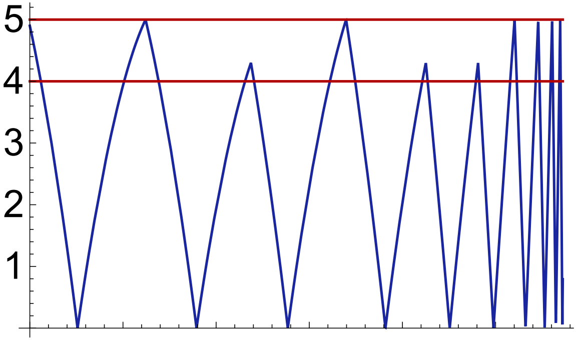
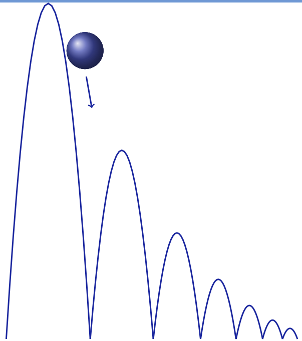
Proof **@invariant**$(2x = 2H - v^2 \wedge x \geq 0 \wedge x{\leq}5)$



Outline

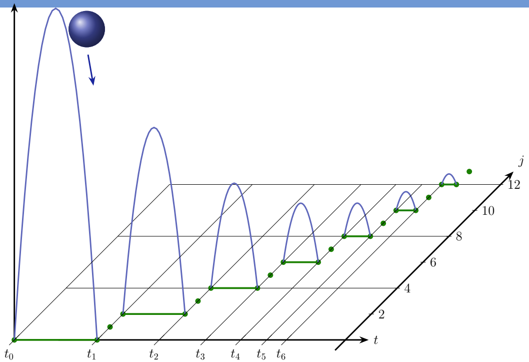
Summary: Time-triggered Control

1. Common paradigm for designing real controllers
2. Periodical or pseudo-periodical control (jitter)
3. Expects delays, expects inertia
4. Implementation: discrete-time sensing
5. Predict events, not just: if(*eventnow*(x)) ...
6. Safe controllers know their own reaction delays
7. Burden of event detection brought to attention of CPS programmer
8. Time-triggered controls are implementable and more robust, but make design and verification more challenging!
9. Use knowledge gained from verified event-triggered model as a basis for designing a time-triggered controller



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$



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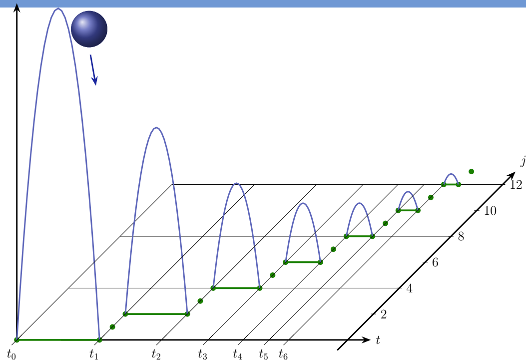

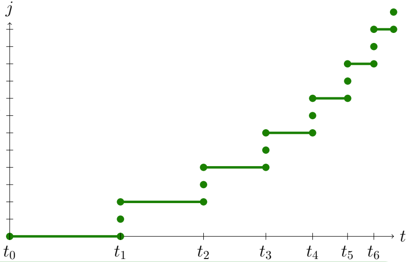

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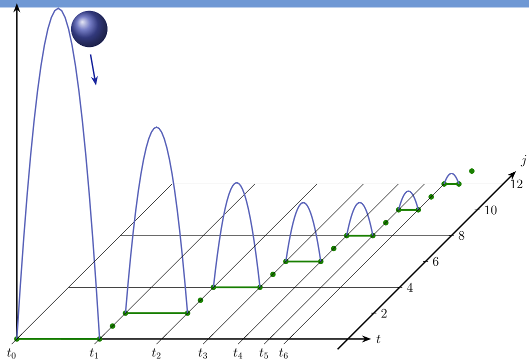
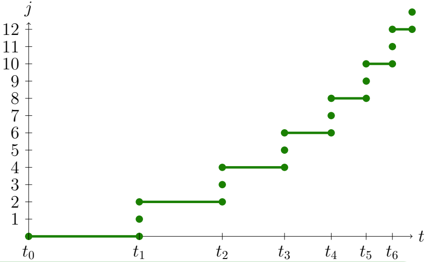
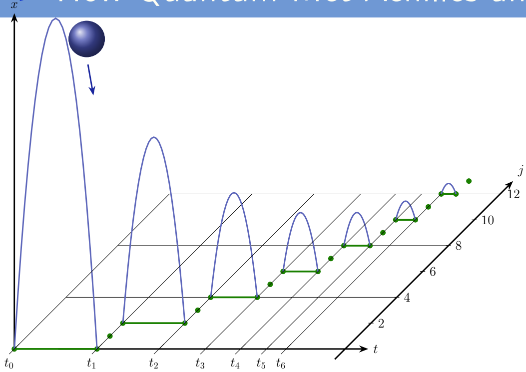
How Quantum Met Achilles and His Tortoise

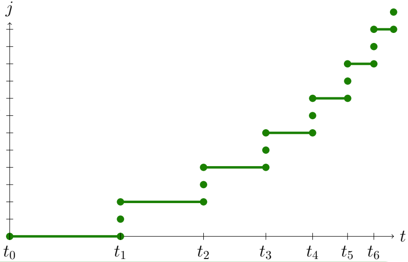




Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

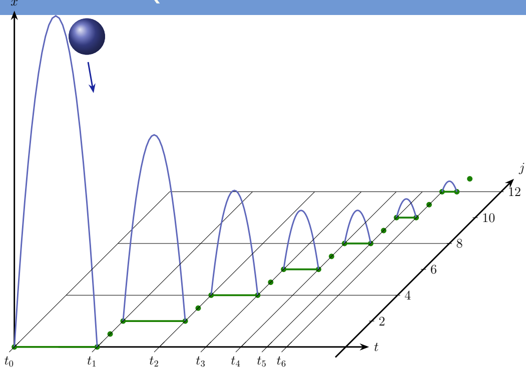


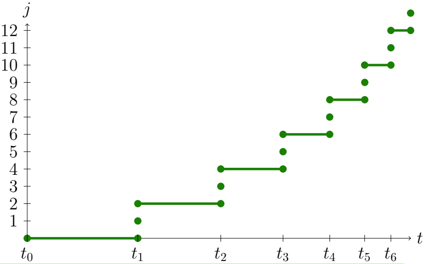


Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i}$$

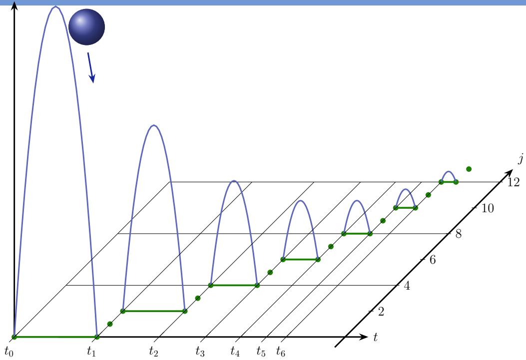
How Quantum Met Achilles and His Tortoise

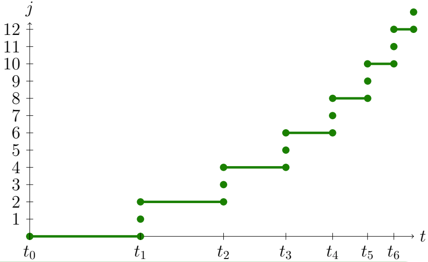




Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}}$$

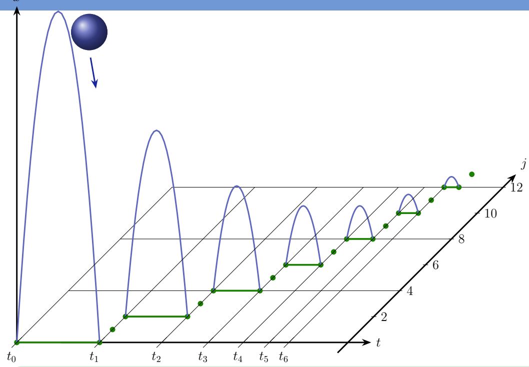


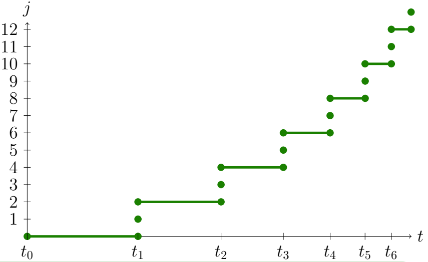


Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2$$

How Quantum Met Achilles and His Tortoise





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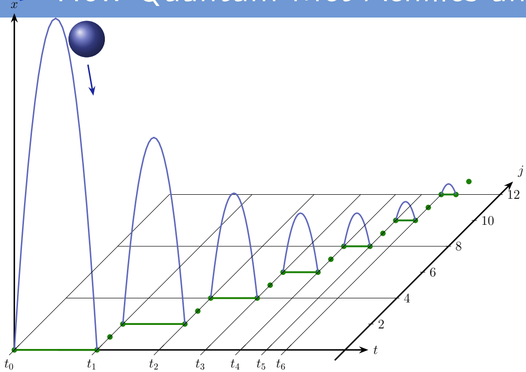

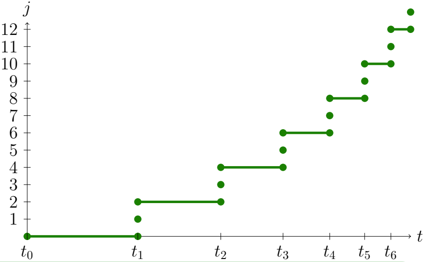

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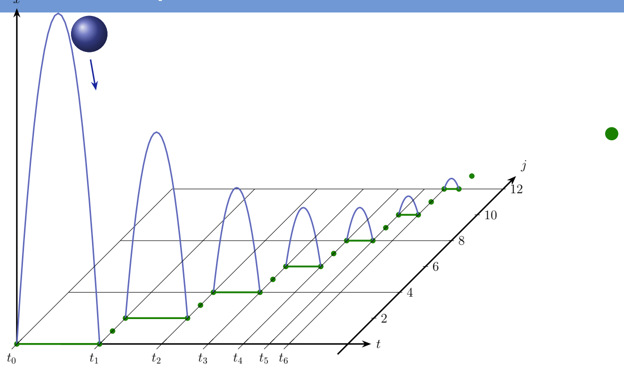
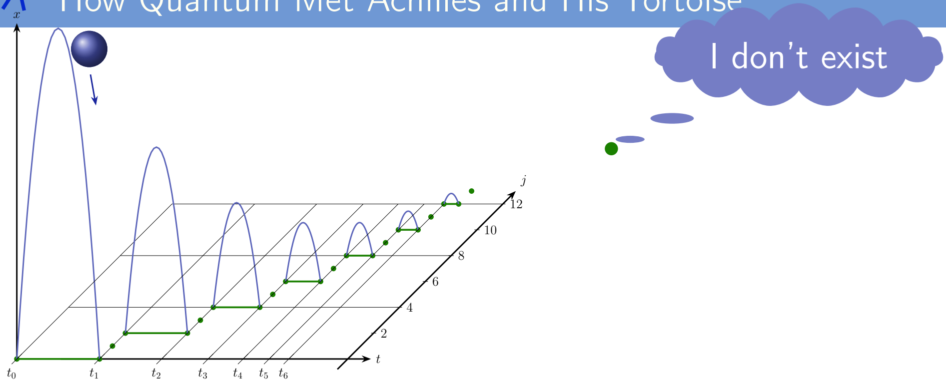
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$



Example (Quantum the Bouncing Ball experiences time)

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\sum_{i=0}^{\infty}\frac{1}{2^i}=\frac{1}{1-\frac{1}{2}}=2<\infty$$



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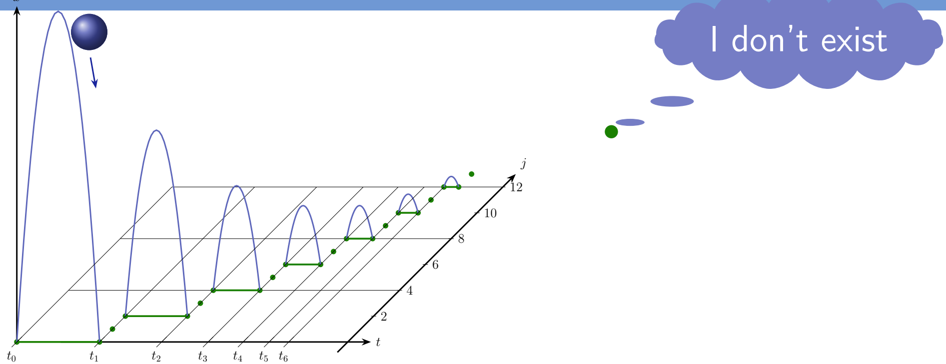

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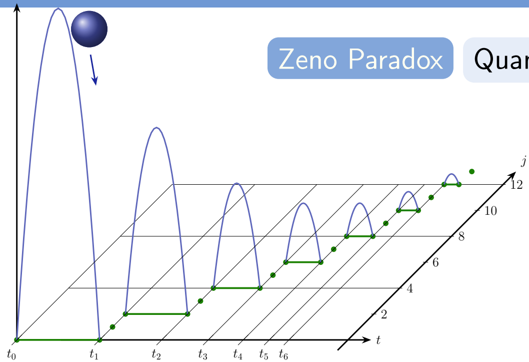
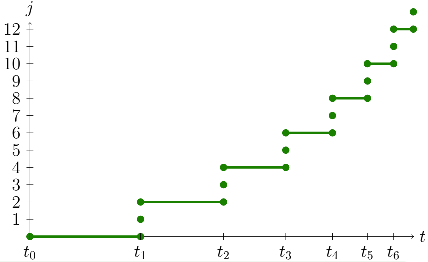


Zeno Paradox Quantum's model causes a time freeze



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