04: Safety & Contracts
Logical Foundations of Cyber-Physical Systems

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Outline

1 Learning Objectives
2 Quantum the Acrophobic Bouncing Ball
3 Contracts for CPS
   - Safety of Robots
   - Safety of Bouncing Balls
4 Logical Formulas for Hybrid Programs
5 Differential Dynamic Logic
   - Syntax
   - Semantics
   - Notational Convention
6 Identifying Requirements of a CPS
7 Summary
Outline

1. Learning Objectives
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6. Identifying Requirements of a CPS
7. Summary
Learning Objectives

Safety & Contracts

- Rigorous specification
- Preconditions
- Postconditions
- Contracts
- Differential dynamic logic

- Analytic specification
- Discrete+continuous
- Reasoning principles

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LFCPS / 04: Safety & Contracts
Learning Objectives

Quantum the Acrophobic Bouncing Ball

Contracts for CPS
  - Safety of Robots
  - Safety of Bouncing Balls

Logical Formulas for Hybrid Programs

Differential Dynamic Logic
  - Syntax
  - Semantics
  - Notational Convention

Identifying Requirements of a CPS

Summary
Example (Quantum the Bouncing Ball)

\[ \begin{align*}
  x' &= v, \\
  v' &= -g
\end{align*} \]
Example (Quantum the Bouncing Ball)

\[ \{ x' = v, v' = -g \} \]
Example (Quantum the Bouncing Ball)
\[
\{ x' = v, \, v' = -g \}
\]
Quantum the Acrophobic Bouncing Ball

Example (Quantum the Bouncing Ball)

\[ \{ x' = v, \: v' = -g \: & \: x \geq 0 \} \]
Example (Quantum the Bouncing Ball)

\[
\begin{align*}
    \{ & x' = v, \\ & v' = -g & x \geq 0 \}; \\
    \text{if} & (x = 0) & v := -cv
\end{align*}
\]
Example (Quantum the Bouncing Ball)

\[
\{ x' = v, \quad v' = -g \quad \& \quad x \geq 0 \};
\]

\[
\text{if}(x = 0) \quad v := -cv
\]
Example (Quantum the Bouncing Ball)

\[
\begin{align*}
\{ & x' = v, \ v' = -g \ & \& x \geq 0; \\
\text{if} & (x = 0) \ v := -cv \}
\end{align*}
\]
Example (Quantum the Bouncing Ball)

\[ \{ x' = v, v' = -g & x \geq 0 \}; \]
\[ \text{if}(x = 0) \quad v := -cv \]

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Example (Quantum the Bouncing Ball)

\[
\begin{align*}
\{ x' &= v, \\
& v' = -g \ \& \ x \geq 0 \}; \\
\text{if}(x = 0) \ v &= -cv
\end{align*}
\]
Example (Quantum the Bouncing Ball)

\[
\{ x' = v, v' = -g \& x \geq 0 \}; \\
\text{if}(x = 0) \ v := -cv
\]
Example (Quantum the Bouncing Ball)

\[
\begin{align*}
\{x' = v, v' = -g & \land x \geq 0 \}; \\
\text{if}(x = 0) (v := -cv \cup v := 0)\nonumber
\end{align*}
\]
Example (Quantum the Bouncing Ball)

\[ \{x' = v, v' = -g \land x \geq 0\}; \]

if \(x = 0\) \(v := -cv\)\(^*\)
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Safety of Robots

Three Laws of Robotics
Isaac Asimov (1942)

1. A robot may not injure a human being or, through inaction, allow a human being to come to harm.
2. A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.
3. A robot must protect its own existence as long as such protection does not conflict with the First or Second Law.

Three Laws of Robotics are not the answer.
They are the inspiration!

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### Three Laws of Robotics

**Isaac Asimov 1942**

1. A robot may not injure a human being or, through inaction, allow a human being to come to harm.

2. A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.

3. A robot must protect its own existence as long as such protection does not conflict with the First or Second Law.
### Three Laws of Robotics

<table>
<thead>
<tr>
<th>Number</th>
<th>Law Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>A robot may not injure a human being or, through inaction, allow a human being to come to harm.</td>
</tr>
<tr>
<td>2.</td>
<td>A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

Three Laws of Robotics are not the answer. They are the inspiration!
Example (Quantum the Bouncing Ball)

\[ \{x' = v, v' = -g \land x \geq 0\}; \]
\[ \text{if}(x = 0) \ v := -cv \]
Example (Quantum the Bouncing Ball)

\textbf{ensures}(0 \leq x)

\[
\{x' = v, v' = -g \& x \geq 0\};
\]
\[
\text{if}(x = 0) \ v := -cv
\]
Example (Quantum the Bouncing Ball)

\[ \text{ensures}(0 \leq x) \]
\[ \text{ensures}(x \leq H) \]
\[ (\{x' = v, v' = -g \land x \geq 0\}; \]
\[ \text{if}(x = 0) v := -cv)^* \]
Example (Quantum the Bouncing Ball)

requires \((x = H)\)

\[\text{ensures} (0 \leq x)\]
\[\text{ensures} (x \leq H)\]

\[\{x' = v, v' = -g & x \geq 0\};\]
\[\text{if}(x = 0) \ v := -cv\] *
Example (Quantum the Bouncing Ball)

\begin{align*}
\text{requires} & \ (x = H) \\
\text{requires} & \ (0 \leq H) \\
\text{ensures} & \ (0 \leq x) \\
\text{ensures} & \ (x \leq H) \\
\{ & \ x' = v, \ v' = -g \land x \geq 0 \} ; \\
& \text{if} (x = 0) \ v := -cv 
\end{align*}
Example (Quantum the Bouncing Ball)

requires \( x = H \)

requires \( 0 \leq H \)

ensures \( 0 \leq x \)

ensures \( x \leq H \)

\[ \{ x' = v, \ v' = -g \land x \geq 0 \} \]

\[ \text{if}(x = 0) \ v := -cv \]

*invariant* \( x \geq 0 \)
Example (Quantum the Bouncing Ball)

\begin{align*}
\text{requires}(x = H) \\
\text{requires}(0 \leq H) \\
\text{ensures}(0 \leq x) \\
\text{ensures}(x \leq H) \\
(x' = v, v' = -g & x \geq 0); \\
\text{if}(x = 0) v := -cv) \quad @\text{invariant}(x \geq 0)
\end{align*}
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Contracts are Not Enough

CPS contracts are crucial for CPS safety. We need to understand CPS programs and contracts and how we can convince ourselves that a CPS program respects its contract.

Contracts are at a disadvantage compared to full logic.

Logic is for Specification and Reasoning

1. Specification of a whole CPS program.
2. Analytic inspection of its parts.
3. Argumentative relations between contracts and program parts. “Yes, this CPS program meets its contract, and here’s why . . .”
Example (Quantum the Bouncing Ball)

- \( \text{requires}(x = H) \)
- \( \text{requires}(0 \leq H) \)
- \( \text{ensures}(0 \leq x) \)
- \( \text{ensures}(x \leq H) \)

\[
\begin{align*}
\{x' = v, v' = -g & \land x \geq 0 \}; \\
\text{if}(x = 0) & v := -cv
\end{align*}
\]
Contracts for Quantum the Acrophobic Bouncing Ball

Example (Quantum the Bouncing Ball)

requires \( x = H \)

requires \( 0 \leq H \)

ensures \( 0 \leq x \)

ensures \( x \leq H \)

\( \{ x' = v, v' = -g & x \geq 0 \}; \)

\( \text{if}(x = 0) \ v := -cv \) *

Precondition: \( x = H \land 0 \leq H \) in FOL

\( x = H \land 0 \leq H \) in FOL
Example (Quantum the Bouncing Ball)

- **requires**($x = H$)
- **requires**($0 \leq H$)
- **ensures**($0 \leq x$)
- **ensures**($x \leq H$)

\[
\{ x' = v, \quad v' = -g \land x \geq 0 \}; \\
\text{if}(x = 0) \ v := -cv
\]

**Precondition:**

\[x = H \land 0 \leq H\] in FOL

**Postcondition:**

\[0 \leq x \land x \leq H\] in FOL
Example (Quantum the Bouncing Ball)

- requires($x = H$)
- requires($0 \leq H$)
- ensures($0 \leq x$)
- ensures($x \leq H$)
- ($\{ x' = v, v' = -g \& x \geq 0 \}$; if($x = 0$) $v := -cv$)*

Precondition: $x = H \land 0 \leq H$ in FOL

Postcondition: $0 \leq x \land x \leq H$ in FOL

How to say post is true after all HP runs?
Example (Quantum the Bouncing Ball)

\begin{align*}
\text{requires}(x = H) & \quad \text{Precondition:} \quad x = H \land 0 \leq H \text{ in FOL} \\
\text{requires}(0 \leq H) & \quad \text{Postcondition:} \quad 0 \leq x \land x \leq H \text{ in FOL} \\
\text{ensures}(0 \leq x) & \\
\text{ensures}(x \leq H) & \\
\{x' = v, \quad v' = -g \land x \geq 0\}; & \\
\text{if}(x = 0) \quad v := -cv &
\end{align*}
Example (Quantum the Bouncing Ball)

\begin{align*}
\text{requires}(x = H) & \quad \text{Precondition:} \quad x = H \land 0 \leq H \text{ in FOL} \\
\text{requires}(0 \leq H) & \quad \text{Postcondition:} \quad 0 \leq x \land x \leq H \text{ in FOL} \\
\text{ensures}(0 \leq x) & \\
\text{ensures}(x \leq H) & \\
\{x' = v, v' = -g \land x \geq 0 \}; & \\
\text{if}(x = 0) v := -cv & \quad (*)
\end{align*}
Example (Quantum the Bouncing Ball)

- **requires**$(x = H)$
- **requires**$(0 \leq H)$
- **ensures**$(0 \leq x)$
- **ensures**$(x \leq H)$

$(\{x' = v, v' = -g & x \geq 0\};$

if$(x = 0) v := -cv)$*

---

**Precondition:**

$x = H \land 0 \leq H$ in FOL

**Postcondition:**

$0 \leq x \land x \leq H$ in FOL

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Example (Quantum the Bouncing Ball)

requires\((x = H)\)

requires\((0 \leq H)\)

ensures\((0 \leq x)\)

ensures\((x \leq H)\)

\[\{x' = v, v' = -g \& x \geq 0\};\]

\[\text{if}(x = 0) v := -cv\]^* 

Precondition:
\[x = H \land 0 \leq H\] in FOL

Postcondition:
\[0 \leq x \land x \leq H\] in FOL
Contracts for Quantum the Acrophobic Bouncing Ball

\[
[(\{x' = v, v' = -g \land x \geq 0\}; \text{if}(x = 0) v := -cv)^*]
\]

Example (Quantum the Bouncing Ball)

- **requires** \(x = H\)
- **requires** \(0 \leq H\)
- **ensures** \(0 \leq x\)
- **ensures** \(x \leq H\)

\(\{x' = v, v' = -g \land x \geq 0\}; \text{if}(x = 0) v := -cv)^*\)

Precondition:
\(x = H \land 0 \leq H\) in FOL

Postcondition:
\(0 \leq x \land x \leq H\) in FOL
[\left( \{ x' = v, v' = -g \land x \geq 0 \}; \text{if}(x=0) v := -cv \right)^*](x \leq H)
Contracts for Quantum the Acrophobic Bouncing Ball

\[(\{x' = v, v' = -g \& x \geq 0\}; \text{if}(x=0) v := -cv)^*](0 \leq x)\]
\[\[(\{x' = v, v' = -g \& x \geq 0\}; \text{if}(x=0) v := -cv)^*](x \leq H)\]

Example (Quantum the Bouncing Ball)

- \textbf{requires}(x = H)
- \textbf{requires}(0 \leq H)
- \textbf{ensures}(0 \leq x)
- \textbf{ensures}(x \leq H)

\((\{x' = v, v' = -g \& x \geq 0\}; \text{if}(x = 0) v := -cv)^*\)
Example (Quantum the Bouncing Ball)

\[
\text{requires}(x = H)
\]

\[
\text{requires}(0 \leq H)
\]

\[
\text{ensures}(0 \leq x)
\]

\[
\text{ensures}(x \leq H)
\]

\[
(\{x' = v, \nu' = -g \& x \geq 0\}; \text{if}(x=0) \nu := -cv)^*[0 \leq x \land x \leq H]
\]

Precondition:

\[x = H \land 0 \leq H \text{ in FOL}\]

Postcondition:

\[0 \leq x \land x \leq H \text{ in FOL}\]
Example (Quantum the Bouncing Ball)

**Precondition:**

\[ x = H \land 0 \leq H \text{ in FOL} \]

**Postcondition:**

\[ 0 \leq x \land x \leq H \text{ in FOL} \]

\[
[(\{x' = v, v' = -g \land x \geq 0\}; \text{if}(x=0) v := -cv)^*](0 \leq x) \land
[(\{x' = v, v' = -g \land x \geq 0\}; \text{if}(x=0) v := -cv)^*](x \leq H) \leftrightarrow
[(\{x' = v, v' = -g \land x \geq 0\}; \text{if}(x=0) v := -cv)^*](0 \leq x \land x \leq H)
\]
Example (Quantum the Bouncing Ball)

**requires**($x = H$)  
**requires**($0 \leq H$)  
**ensures**($0 \leq x$)  
**ensures**($x \leq H$)  

\[
([\{x' = v, v' = -g \& x \geq 0\}; \text{if}(x=0) v := -cv]^*) (0 \leq x)
\]

**Precondition:**  
$x = H \land 0 \leq H$ in FOL

**Postcondition:**  
$0 \leq x \land x \leq H$ in FOL
Example (Quantum the Bouncing Ball)

**requires**($x = H$)

**requires**($0 \leq H$)

**ensures**($0 \leq x$)

**ensures**($x \leq H$)

($\{x' = v, v' = -g \& x \geq 0\}; \text{if}(x=0)\ v := -cv)^*$

Precondition:

$x = H \land 0 \leq H$ in FOL

Postcondition:

$0 \leq x \land x \leq H$ in FOL
Example (Quantum the Bouncing Ball)

requires(x = H)

requires(0 ≤ H)

ensures(0 ≤ x)

ensures(x ≤ H)

\( \{ x' = v, v' = -g \& x \geq 0 \}; \text{if}(x=0) v := -cv \) *

Precondition:

\( x = H \land 0 \leq H \) in FOL

Postcondition:

\( 0 \leq x \land x \leq H \) in FOL
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Definition (Syntax of differential dynamic logic)

The formulas of differential dynamic logic are defined by the grammar:

\[ P, Q ::= e \geq \bar{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid \forall x P \mid \exists x P \mid [\alpha]P \mid \langle \alpha \rangle P \]
Definition (Syntax of differential dynamic logic)

The formulas of differential dynamic logic are defined by the grammar:

\[ P, Q ::= e \geq \bar{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \rightarrow Q \mid \forall x P \mid \exists x P \mid [\alpha]P \mid \langle \alpha \rangle P \]

- Not
- And
- Or
- Imply
- All reals
- Some real
- All runs
- Some runs
Definition (dL Formulas)

\[ \omega \]

\[ [\alpha]P \]

\[ P \]

\[ P \]

\[ P \]
Definition (dL Formulas)

\[ \langle \alpha \rangle P \]
Definition (dL Formulas)

\[ [\alpha]P \]

\( \omega \) \(-\)span

\( \alpha \)-span
Definition (dL Formulas)

\[ [\alpha]P \]

\[ \langle \beta \rangle P \]

\[ \omega \]

\[ \beta\text{-span} \]

\[ \alpha\text{-span} \]
Definition (dL Formulas)

\( \langle \beta \rangle P \)

\( [\alpha] P \)

\( \omega \)

\( \beta \) - span

\( \alpha \) - span
**Definition (Syntax of differential dynamic logic)**

The formulas of differential dynamic logic are defined by the grammar:

\[ P, Q ::= e \geq \bar{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \to Q \mid \forall x P \mid \exists x P \mid [\alpha]P \mid \langle \alpha \rangle P \]

**Definition (dL semantics)**

\[
\begin{align*}
[e \geq \bar{e}] &= \{\omega : \omega[e] \geq \omega[\bar{e}]\} \\
[P \land Q] &= [P] \cap [Q] \\
[P \lor Q] &= [P] \cup [Q] \\
[P \to Q] &= [P]^C \cup [Q] \\
[\langle \alpha \rangle P] &= [\alpha] \circ [P] = \{\omega : \nu \in [P] \text{ for some } \nu : (\omega, \nu) \in [\alpha]\} \\
[[\alpha]P] &= [\neg \langle \alpha \rangle \neg P] = \{\omega : \nu \in [P] \text{ for all } \nu : (\omega, \nu) \in [\alpha]\} \\
[\exists x P] &= \{\omega : \omega_x^r \in [P] \text{ for some } r \in \mathbb{R}\} \\
[\forall x P] &= \{\omega : \omega_x^r \in [P] \text{ for all } r \in \mathbb{R}\} \\
\omega_x^d(y) &= \begin{cases} d & \text{if } y=x \\ \omega(y) & \text{if } y \neq x \end{cases}
\end{align*}
\]
\([P]\) the set of states in which formula \(P\) is true
\(\omega \in [P]\) formula \(P\) is true in state \(\omega\), alias \(\omega \models P\)
\(\models P\) formula \(P\) is valid, i.e., true in all states \(\omega\), i.e., \([P] = S\)

**Definition (dL semantics)**

\[
\begin{align*}
[\mathcal{e} \geq \mathcal{e}'] &= \{\omega : \omega[\mathcal{e}] \geq \omega[\mathcal{e}']\} \\
[P \land Q] &= [P] \cap [Q] \\
[P \lor Q] &= [P] \cup [Q] \\
[P \rightarrow Q] &= [P]^c \cup [Q] \\
[\langle \alpha \rangle P] &= [\alpha] \circ [P] = \{\omega : \nu \in [P] \text{ for some } \nu : (\omega, \nu) \in [\alpha]\} \\
[[\alpha] P] &= [-\langle \alpha \rangle \neg P] = \{\omega : \nu \in [P] \text{ for all } \nu : (\omega, \nu) \in [\alpha]\} \\
[\exists x P] &= \{\omega : \omega^r_x \in [P] \text{ for some } r \in \mathbb{R}\} \\
[\forall x P] &= \{\omega : \omega^r_x \in [P] \text{ for all } r \in \mathbb{R}\}
\end{align*}
\]
Differential Dynamic Logic: Syntax & Semantics

\[[P]\] the set of states in which formula \(P\) is true
\(\omega \in [P]\) formula \(P\) is true in state \(\omega\), alias \(\omega \models P\)
\(\models P\) formula \(P\) is valid, i.e., true in all states \(\omega\), i.e., \([P]\) = \(S\)

\[\exists d \ [x := 1; x' = d] x \geq 0\] and \[\ [x := x + 1; x' = d] x \geq 0\] and \(\langle x' = d\rangle x \geq 0\)

**Definition (dL semantics)**

\[[e \geq \bar{e}]\] = \(\{\omega : \omega[e] \geq \omega[\bar{e}]\}\)

\[[\neg P]\] = \([P]\)\(^C\) = \(S \setminus [P]\)

\[[P \land Q]\] = \([P]\) \cap \([Q]\)

\[[P \lor Q]\] = \([P]\) \cup \([Q]\)

\[[P \rightarrow Q]\] = \([P]\)\(^C\) \cup \([Q]\)

\[[\langle\alpha\rangle P]\] = \[[\alpha]\] \circ \([P]\) = \(\{\omega : \nu \in [P] \text{ for some } \nu : (\omega, \nu) \in [\alpha]\}\)

\[[[\alpha]P]\] = \[[\neg\langle\alpha\rangle\neg P]\] = \(\{\omega : \nu \in [P] \text{ for all } \nu : (\omega, \nu) \in [\alpha]\}\)

\[[\exists x \ P]\] = \(\{\omega : \omega^r_x \in [P] \text{ for some } r \in \mathbb{R}\}\)

\[[\forall x \ P]\] = \(\{\omega : \omega^r_x \in [P] \text{ for all } r \in \mathbb{R}\}\)
Differential Dynamic Logic: Syntax & Semantics

- \([P]\) the set of states in which formula \(P\) is true
- \(\omega \in [P]\) formula \(P\) is true in state \(\omega\), alias \(\omega \models P\)
- \(\models P\) formula \(P\) is valid, i.e., true in all states \(\omega\), i.e., \([P] = S\)

\(\models \exists d \ [x := 1; x' = d] x \geq 0\) and \(\not\models [x := x + 1; x' = d] x \geq 0\) and \(\not\models \langle x' = d \rangle x \geq 0\)

**Definition (dL semantics)**

- \([e \geq \tilde{e}]\) = \(\{\omega : \omega[e] \geq \omega[\tilde{e}]\}\)
- \([-P]\) = \([P]^c = S \setminus [P]\)
- \([P \land Q]\) = \([P] \cap [Q]\)
- \([P \lor Q]\) = \([P] \cup [Q]\)
- \([P \rightarrow Q]\) = \([P]^c \cup [Q]\)
- \([\langle \alpha \rangle P]\) = \([\alpha] \circ [P] = \{\omega : \nu \in [P] \text{ for some } \nu : (\omega, \nu) \in [\alpha]\}\)
- \([\lnot \langle \alpha \rangle \lnot P]\) = \(\{\omega : \nu \in [P] \text{ for all } \nu : (\omega, \nu) \in [\alpha]\}\)
- \(\exists x P\) = \(\{\omega : \omega^r_x \in [P] \text{ for some } r \in \mathbb{R}\}\)
- \(\forall x P\) = \(\{\omega : \omega^r_x \in [P] \text{ for all } r \in \mathbb{R}\}\)
Notational Conventions: Precedence

Convention (Operator Precedence)

1. **Unary operators** (e.g., *, ¬, ∀x, ∃x, [α], ⟨α⟩) bind stronger than binary
2. ∧ binds stronger than ∨, which binds stronger than →, ↔
3. ; binds stronger than ∪
4. Arithmetic operators +, −, · associate to the left
5. Logical and program operators associate to the right

Example (Operator Precedence)

\[
\begin{align*}
[\alpha]P \land Q & \equiv ([\alpha]P) \land Q \\
\forall x P \land Q & \equiv (\forall x P) \land Q \\
\forall x P \rightarrow Q & \equiv (\forall x P) \rightarrow Q \\
\alpha ; \beta \cup \gamma & \equiv (\alpha ; \beta) \cup \gamma \\
\alpha \cup \beta ; \gamma & \equiv \alpha \cup (\beta ; \gamma) \\
\alpha ; \beta^* & \equiv \alpha ; (\beta^*) \\
P \rightarrow Q \rightarrow R & \equiv P \rightarrow (Q \rightarrow R).
\end{align*}
\]

But →, ↔ expect explicit parentheses. Illegal: \( P \rightarrow Q \leftrightarrow R \quad P \leftrightarrow Q \rightarrow R \)
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Example (Bouncing Ball)

\[
\{x' = v, \quad v' = -g \quad \& \quad x \geq 0\};
\]
\[
\text{if}(x = 0) \quad v := -cv
\]
Example (Bouncing Ball)

\[ H = x \geq 0 \quad \rightarrow \quad \left[ \{ x' = v, v' = -g \& x \geq 0 \}; \right. \]

\[ \left. \text{if}(x = 0) \; v := -cv \right) * \quad 0 \leq x \leq H \]
Quantum the Acrophobic Bouncing Ball

Not if \( g < 0 \) in anti-gravity

Example (Bouncing Ball)

\[
H = x \geq 0 \\
\rightarrow [(\{ x' = v, v' = -g \wedge x \geq 0 \}; \text{if}(x = 0) v := -cv)^*] 0 \leq x \leq H
\]
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

\[ H = x \geq 0 \land g > 0 \rightarrow \left[ \{ x' = v, v' = -g \land x \geq 0 \}; \right.

\left. \text{if } (x = 0) \text{ } v := -cv \right] \] \quad 0 \leq x \leq H
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

\[ H = x \geq 0 \land g > 0 \rightarrow \left[ \{ x' = v, v' = -g \land x \geq 0 \} ; \\
\text{if} (x = 0) v := -cv \right]^* \] 0 \leq x \leq H

Not if \( c > 1 \) for anti-damping
Example (Bouncing Ball)

$$1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[ \left\{ x' = v, v' = -g \land x \geq 0 \right\}; \right. $$

if \( x = 0 \) \( v := -cv \) \left. \right] 0 \leq x \leq H$$
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

\[ v = \begin{cases} 0 & \text{if } x = 0 \text{ and } v < 0 \\ -cv & \text{if } x = 0 \text{ and } v > 0 \\ v & \text{otherwise} \end{cases} \]

\[ v' = -g \land x \geq 0 \]

\[ x' = v \]

\[ 0 \leq x \leq H \]

Not if \( v > 0 \) initial climbing
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

$$\nu \leq 0 \land 1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[ \{ x' = \nu, v' = -g \land x \geq 0 \}; \text{if } (x = 0) \; \nu := -cv \right]^{*} 0 \leq x \leq H$$
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

\[ v \leq 0 \land 1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[ \{ x' = v, v' = -g \land x \geq 0 \}; \right. \\
\left. \text{if}(x = 0) v : = -cv \right]^{*} \quad 0 \leq x \leq H \]
Quantum the Acrophobic Bouncing Ball

Example (Bouncing Ball)

\[ v = 0 \land 1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \left[ \begin{array}{l}
\{ x' = v, v' = -g \land x \geq 0 \}; \\
\text{if } (x = 0) \; v := -cv \end{array} \right] 0 \leq x \leq H \]
Example (Runaround Robot)

\[
Q \equiv (x + w - o_x)^2 + (y - v - o_y)^2 \neq v^2 + w^2
\]

Obstacle not on tangential circle

Obstacle not on ray \((x, y) + R(v, w)\)

\[
\omega \quad (x, y) \quad (v, w)
\]

\[
\omega : \{-1 \cup \omega \equiv 1 \cup \omega \equiv 0 \}
\]

\[
\begin{align*}
(x') &= v, \\
y' &= w, \\
v' &= \omega w, \\
w' &= -\omega v
\end{align*}
\]

\[
(x, y) \neq o \rightarrow \left[ (\omega : -1) \cup \omega : 1 \cup \omega : 0 \right]
\]
Example (Runaround Robot)

\[
((\omega := -1 \cup \omega := 1 \cup \omega := 0); \\
\{x' = v, y' = w, v' = \omega w, w' = -\omega v\})^*
\]
Example (Runaround Robot)

\((x, y) \neq o \rightarrow [(ω := \{-1\} \cup ω := \{1\} \cup ω := \{0\});
\{x' = v, y' = w, v' = ωw, w' = -ωv\}]^{*} (x, y) \neq o\)
Example (Runaround Robot)

\[(x, y) \neq o \rightarrow [((?Q_{-1}; \omega := -1 \cup ?Q_1; \omega := 1 \cup ?Q_0; \omega := 0); \\
\{x' = v, y' = w, v' = \omega w, w' = -\omega v\})^*] (x, y) \neq o\]
Outline

1. Learning Objectives
2. Quantum the Acrophobic Bouncing Ball
3. Contracts for CPS
   - Safety of Robots
   - Safety of Bouncing Balls
4. Logical Formulas for Hybrid Programs
5. Differential Dynamic Logic
   - Syntax
   - Semantics
   - Notational Convention
6. Identifying Requirements of a CPS
7. Summary
### Definition (Hybrid program $\alpha$)

$$
\alpha, \beta ::= x := f(x) \mid ?Q \mid x' = f(x) & Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*
$$

### Definition (dL Formula $P$)

$$
P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid \forall x \ P \mid \exists x \ P \mid [\alpha]P \mid \langle \alpha \rangle P
$$
Differential Dynamic Logic dL: Syntax

Definition (Hybrid program $\alpha$)

$$\alpha, \beta ::= x := f(x) \mid ?Q \mid x' = f(x) \& Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula $P$)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x \ P \mid \exists x \ P \mid [\alpha]P \mid \langle\alpha\rangle P$$
Differential Dynamic Logic dL: Semantics

**Definition (Hybrid program semantics)**

\[
\begin{align*}
[x := f(x)] &= \{(ω, ν) : ν = ω \text{ except } ν[x] = ω[f(x)]\} \\
[?Q] &= \{(ω, ω) : ω ∈ [Q]\} \\
[x' = f(x)] &= \{(φ(0), φ(r)) : φ \models x' = f(x) \text{ for some duration } r\} \\
[α ∪ β] &= [α] ∪ [β] \\
[α; β] &= [α] ◦ [β] \\
[α^*] &= [α]^* = \bigcup_{n ∈ \mathbb{N}} [α^n] \\
\end{align*}
\]

**Definition (dL semantics)**

\[
\begin{align*}
[e ≥ ẽ] &= \{ω : ω[e] ≥ ω[ẽ]\} \\
[¬P] &= [P]^C \\
[P ∧ Q] &= [P] ∩ [Q] \\
[⟨α⟩P] &= [α] ◦ [P] = \{ω : ν ∈ [P] \text{ for some } ν : (ω, ν) ∈ [α]\} \\
[[α]P] &= [¬⟨α⟩¬P] = \{ω : ν ∈ [P] \text{ for all } ν : (ω, ν) ∈ [α]\} \\
[∃x P] &= \{ω : ω^r_x ∈ [P] \text{ for some } r ∈ \mathbb{R}\} \\
\end{align*}
\]
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